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MANIPULATIVE MATERIALS IN MATHEMATICS EDUCATION

MATERIAIS MANIPULATIVOS EM EDUCAÇÃO MATEMÁTICA: CATEGORIZAÇÃO, USOS E EQUÍVOCOS

MATERIALES MANIPULATIVOS EN LA ENSEÑANZA DE LAS MATEMÁTICAS

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ABSTRACT

The aim of this text is to present an understanding of manipulative materials in mathematics education and to discuss their uses, care, and mistakes in pedagogical practices. These materials in the field of education encompass any physical, pictorial, or virtual objects used as resources for teaching specific disciplinary content have been grouped into three categories: didactically constructed materials, which include all types of material artificially created by educators to simulate mathematical objects and relationships that stimulate the construction of mathematical ideas; cultural instruments inherited from tradition, which have accompanied and helped the theoretical development of mathematics; and objects taken from everyday life, which attest, in some way, to some fragment of mathematical knowledge. Distinctions are made between the materials used to teach numbers and operations regarding how they take on values and how they are structured, as well as the closeness or distance between games and manipulative materials. The pros and cons of using manipulative materials in mathematics education are also discussed, emphasizing possible misconceptions regarding the production or use of these materials. Finally, it was concluded that the efficiency and effectiveness of the use of manipulatives seem to be related to three variables: the choice of material, clear and participatory instruction from the teacher, and participation in the use of the material by students through a mathematical process in which the teacher and their students make and attribute meanings to the manipulative objects.

Keywords: *Mathematical relationships; School mathematics; Games; Number system; Teaching and learning.*

RESUMO

O objetivo deste texto é apresentar uma compreensão acerca de materiais manipulativos na Educação Matemática, bem como discutir seus usos, cuidados e equívocos nas práticas pedagógicas. Esses materiais, que no campo educacional englobam quaisquer objetos físicos, pictóricos ou virtuais utilizados como recursos para o ensino de determinado conhecimento, foram agrupados em três categorias: materiais didaticamente construídos, que incluem todo tipo de material criado artificialmente por educadores para simular objetos e relações matemáticas que estimulem a construção de ideias matemáticas; instrumentos culturais herdados da tradição, que acompanharam e auxiliaram o desenvolvimento teórico da matemática; e objetos retirados da vida cotidiana, que atestam, de certa forma, algum fragmento do conhecimento matemático. São apresentadas distinções entre materiais utilizados no ensino de números e operações, em se tratando da forma como assumem valores e a sua estruturação, além de uma aproximação ou distanciamento entre jogos e materiais manipulativos. Os prós e contras acerca da utilização de materiais manipulativos na educação matemática também são discutidos, com

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ênfase a possíveis equívocos quanto à produção ou utilização desses materiais. Por fim, concluiu-se que a eficiência e eficácia do uso de manipulativos parecem estar relacionadas a três variáveis: a escolha do material, a clara e participativa instrução do professor e a participação no uso do material pelos estudantes, por meio de um processo matemático, em que o professor e os seus alunos fazem e atribuem sentidos aos objetos manipulativos.

Palavras-chave: Relações matemáticas; Matemática escolar; Jogos; Sistema de Numeração; Ensino e Aprendizagem.

RESUMEN

El objetivo de este texto es presentar una comprensión de los materiales manipulativos en la Educación Matemática, así como discutir sus usos, cuidados y errores en las prácticas pedagógicas. Estos materiales, que en el ámbito educativo engloban cualquier objeto físico, pictórico o virtual utilizado como recurso para la enseñanza de determinados conocimientos, se agruparon en tres categorías: materiales didácticamente construidos, que incluyen todo tipo de material creado artificialmente por los educadores para simular objetos y relaciones matemáticas que estimulen la construcción de ideas; instrumentos culturales heredados de la tradición, que acompañaron y ayudaron al desarrollo teórico de las matemáticas; y objetos tomados de la vida cotidiana, que dan fe, en cierta manera, de algún fragmento de conocimiento matemático. Se presentan distinciones entre los materiales utilizados en la enseñanza de los números y las operaciones, en cuanto a la forma en que asumen valores y su estructuración, así como una aproximación o distancia entre juegos y materiales manipulativos. También se discuten los pros y los contras del uso de materiales manipulativos en la educación matemática, con énfasis en posibles malentendidos respecto de la producción o uso de estos materiales. Finalmente, se concluyó que la eficiencia y efectividad del uso de manipulativos parecen estar relacionadas con tres variables: la elección del material, la instrucción clara y participativa del docente y la participación en el uso del material por parte de los estudiantes, a través de un proceso matemático, en el que el profesor y sus alumnos crean y atribuyen significados a objetos manipulables.

Palabras-clave: *Relaciones matemáticas; Matemáticas escolares; Juegos; Sistema de numeración; Enseñanza y aprendizaje.*

INTRODUCTION

The nature of school mathematics challenges teachers as much as students. The challenges related to the processes of teaching and learning in school mathematics lead teachers to seek teaching resources that can contribute to a learning process that is more meaningful to students, at different levels and modalities of education. Educators responsible for psycho-pedagogical activities also seek teaching resources that can minimize the difficulties of learning mathematics. There is yet another group of interest, the designers of teaching materials, of textbooks, and the producers of materials, who suggest the use of such resources through textual representation in their textbooks.

Among several resources, especially in what concerns the teaching of mathematics for children in their first school years, manipulatives and their representations are always present in textbook representations as well as physically in the classroom environment. Over the years, assumptions regarding school mathematics education have shown that it is not only children who benefit from the use of manipulatives for the development of mathematical thinking. Over the course of their schooling, students of all ages can benefit from various teaching resources, including manipulatives.

In fact, to really contribute to the learning of mathematics, manipulatives need to be carefully constructed, selected, and utilized for learning purposes. Along these lines, the aim of this text is to present an understanding about manipulatives in mathematics education and also to

VIDYA, v. 45, n. 1, p. 135-152,
jan./jun. - Santa Maria, 2025.

discuss their usages, needed precautions, as well as the misconceptions surrounding them in pedagogical practices.

MANIPULATIVE MATERIALS: A CATEGORIZATION

When it comes to the educational field, manipulative materials[4] include objects, whether physical or pictorial, just as long as they are detached from the page or can be cut out so that they can move from a static position to enjoy the dynamism of manipulation, or virtual, used as resources for teaching a determined knowledge. We contend that because the expression - "manipulatives" - is more comprehensive, it encompasses other terms[5] often used for tactile and multi-sensory resources. We are particularly interested in discussing their use in mathematics education.

Although it is common to think of manipulatives as objects specifically designed for didactic purposes, many of them come from different origins and can be adapted to those ends. Even though they do not make these purposes explicit, Bartolini Bussi and Boni (2003) present a classification that differentiates into four categories[6] what they call "[…] instruments historically used in mathematical experiments or in the teaching tradition" (15). Such classification served as a basis so that, from adaptations, we can differentiate the origin of the most varied types of manipulatives used as resources for the teaching of mathematics in three categories:

> 1. *Materials constructed for didactic purposes*, which include all types of materials artificially created by educators to simulate objects and mathematical relations that stimulate the development of ideas. These materials can be physical, as with the case of Base Ten Blocks, Cuisenaire rods, and the Geoboard; pictorial, such as figures which are cut out from the page to be manipulated, for example, the planification of three-dimensional surfaces; or virtual, with dynamic adaptations of base ten blocks or Cuisenaire rods, enabled by digital technology.
> 2. *Cultural instruments inherited from tradition*, that accompany and aid the theoretical development of mathematics, such as the abacus, the soroban, the ruler, and the compass, considering their original physical form or their virtual adaptations (dynamic geometry software, for example);
> 3. *Objects taken from day-to-day life*, that attest, in some way, some fragment of mathematical knowledge, such as string, coins, toys, sticks, seeds or stones, including their physical, pictorial, cut-out, and virtual and dynamic representations.

Different from what Bartolini Bussi and Boni (2003) proposed, according to our classification system, virtual manipulatives[7] are distributed in the three categories previously mentioned. They are defined by Moyer-Packenham and Bolyard (2016) as visual interactive representations and enabled by technologies of dynamic mathematical objects, including all the programmable resources that allow them to be manipulated, and present opportunities for the construction of mathematical knowledge. For these

4 The term "manipulatives"and "manipulative materials" will be used in this text as synonyms, thus avoiding excessive repetition.

5 Didactic manipulative material, concrete material, concrete manipulatives, manipulative resources, virtual manipulative material, manipulative aids, etc.

6 The categories defined by the authors are: (a) concrete materials, artificially designed by educators; (b) cultural instruments inherited from tradition; (c) technological objects taken from everyday life; and (d) software developed through information technology.

7 According to Moyer-Packenham and Bolyard (2016), the terms "digital manipulatives," "computer manipulatives," and "virtual manipulatives" have been commonly used synonymously.

authors, in the future, "[…] it is very likely that virtual manipulatives are no longer based on any technology (for example, they can project 3D objects or holographic images)" (Moyer-Packenham; Bolyard, 2016, p. 06). We, on the other side, understand that the creation of holographic manipulatives or in amplified or expanded virtual realities, for example, does not preclude the existence of virtual manipulatives based on technologies, as they are currently known and used.

As for pictorial manipulatives included in the didactically built materials, there are important considerations to be made. Base-ten blocks, which are excessively represented in Brazilian textbooks, for instance, may or may not be considered manipulatives. However, static figures in the pages of textbooks are only images, for they lack dynamism, that is, they cannot be physically manipulated. In this case, they are only static illustrations of manipulatives, that can trigger mathematical ideas in the mind of the reader. Some textbook collections, nevertheless, include inserts with pictorial representations of base-ten blocks so that the students can cut them out and manipulate them. These are examples of pictorial manipulatives, according to our first category.

Regarding pictorial manipulatives included in our third category, which refers to objects taken from day-to-day life, we can think in a way analogous to that exposed in the previous paragraph. Static figures in the pages of textbooks should not be considered as manipulatives. They are only static representations of objects that originate from everyday life. On the other hand, when dealing with cut-out figures from books, inserts, and magazines, among others, these figures would represent pictorial manipulatives from objects taken from day-to-day life.

ON MANIPULATIVES TRADITIONALLY UTILIZED TO TEACH NUMBERS AND OPERATIONS

When dealing with non-positional manipulatives, designed to teach numbers and the four basic mathematical operations, Silveira (2021) affirms that they can be divided into two groups: *materials with apparent values* and *materials with conventional values*. Materials with apparent values are those in which the component pieces not only bring their apparent values in their form or grouping, but these values are also organized proportionally, according to powers of ten; that is, a piece that holds the value of ten units is valued ten times more than a single piece that is worth one unit. Base-Ten Blocks, bundles of sticks, Digi-Blocks and Interlocking Cubes are good examples of materials that possess apparent value. Van de Walle *et al*. (2013) name this material category as "proportional models."

Materials with conventional values do not clearly display the value of each component piece. It is dependent on a previously established convention that guarantees that a piece is worth a certain quantity. Green chips[8], play money, colored chips, and abacus rings[9] in different colors are examples of materials with conventional values. Van de Walle *et al*. (2013) considers these materials as "non-proportional models."

Materials with apparent values can be divided in two encompassing groups, according to Van de Walle *et al.* (2013): (a) Groupable Models, and (b) Pregrouped Models. Groupable materials are composed of pieces that always have the value of one unit, that is, they are always worth 10^0. With these units, however, it is possible to create groups of quantities corresponding to other powers of 10. Popsicle sticks are a good example of this type of material. Each popsicle stick always has the value of one unit, but with the help of a rubber band it is possible to join 10 of these sticks to make a "small bundle," that represents a ten. It is still possible to join 10 of these bundles with another rubber

8 Green chips that simulate base-ten, presented and provided in the Ápis Matemática textbook series published by Ática press.
9 Only the rings, without the structure of the abacus.

VIDYA, v. 45, n. 1, p. 135-152, jan./jun. - Santa Maria, 2025.

band to form a “big bundle” that represents a hundred. In the same way, it is possible to “ungroup” these collections by simply removing the rubber band. That would facilitate. It would make it easier in cases of subtraction with regrouping, for example. Among the groupable materials the popsicle sticks, soda straws, Digi-Blocks, Interlocking Cubes and beans stand out.

Pregrouped materials are composed of pieces that originally possess apparent values, purposely matching with different powers of 10. Because of the way they are manufactured, these pieces cannot be ungrouped. Base-ten blocks are examples of pregrouped manipulatives, because they include blocks that are worth one unit (unit cubes), ten blocks or a ten (longs), one hundred units or a hundred (flats), one thousand units or a thousand (cube). When manipulating these materials, it is not possible to take apart a flat to obtain ten longs, or to join ten longs to make a flat. When such actions are necessary, as in addition or subtraction with regrouping (“carry” or “borrow”), the child needs to remove one or ten blocks from the system. In addition, when there is a “carry”, ten blocks of a lower place value are exchanged for one block of a higher place value, representing the same quantity of units. In subtraction, when necessary, the opposite is done: an external exchange of one block for ten blocks is made, and finally, the new quantity of blocks is then reinserted into the system. Paper, cardboard, or card manipulatives used in teaching decimal numbers are also examples of pregrouped materials.

RELATIONS BETWEEN MANIPULATIVES AND GAMES

Moyer-Packenham and Bolyard (2016) describe the relation between virtual manipulatives and games, considering that it occurs when virtual manipulatives are incorporated in gaming environments. From their discussion, we establish approximations and differences between virtual or physical manipulatives and games. In many cases, physical or virtual manipulatives are incorporated as an integral part of a game, without which it would not exist. There are cases in which games are created from a specific manipulative, aiming to make it more engaging and meaningful for students, since the manipulation of the material takes place in a game environment, involving competition and rules associated with the material’s manipulation.

To exemplify, we can think of the game “Never 10”[10]. In the game, the players accumulate pieces of base-ten blocks based on dice rolls and respecting the rule of never accumulating ten pieces of the same type, such as ten cubes or ten longs. This could lead children to become familiar with exchanging groups of ten objects for an object of equivalent value. It is important to stress that not everything in the game can be considered a manipulative material, but often, as in the previous example, the game can only be played by manipulating objects that are embedded within it. We consider that incorporating manipulatives in physical or virtual game environments can foster gains for mathematical learning.

10 A game to explore place value in base-ten learning.

VIDYA, v. 45, n. 1, p. 135-152,
jan./jun. - Santa Maria, 2025.

MANIPULATIVE MATERIALS: A CATEGORIZATION

Researchers in educational psychology, such as Piaget, Gattegno,[11] and Bruner, stand among the most prominent advocates for the use of manipulatives for the teaching and learning of mathematics. In addition to advocating for the use of such resources, Cuisenaire, Gattegno, Dienes, and Montessori developed manipulatives that, with or without modifications, remain widely used, such as George Cuisenaire's rods, Caleb Gattegno's Geoboards, base-ten blocks or Zoltan Dienes's blocks[12], and Maria Montessori's bead chains[13].

Building on the pioneering work of those educators, researchers in mathematics education have dedicated their work to studying the effects of using manipulatives. Ladel and Kortenkamp (2016) understand that the basis for promoting mathematical learning processes lies in operations with manipulatives. For some researchers, these materials may facilitate connections that help students relate their knowledge and informal experience with mathematical abstractions (Kilpatrick *et al.*, 2001), or even help them understand abstract mathematical concepts, which help them to connect concepts and informal, intuitive, concrete ideas (Uribe-Flórez; Wilkins, 2010).

Manipulatives allow invisible mathematical concepts to become visible (Golafshani, 2013), or make mathematics more real for students (Baroody, 1989). For Moyer-Packenham and Jones (2004), these materials also aid those students who have difficulty understanding abstract symbols. Furner *et al*. (2005) understand that the use of manipulatives helps students build connections between the concrete and the abstract. From this perspective, there is a misconception that it is possible to conceive of concrete mathematics. In fact, the connections are established between the concreteness of the material being manipulated and the mathematical relations, which are abstract. It is important to emphasize that concreteness is not simply related to the object itself being concrete, but to the meaning it carries, even if it is virtual, for example.

Notably, Uttal *et al.* (1997) argue that it can be counterproductive to use manipulative materials that do not allow students to understand the relation between the material and the mathematical concept being taught. Without this understanding, children can think they are learning two different things at the same time, feeling cognitively overloaded. From the same perspective, Brown *et al.* (2009) understand that when using manipulatives, students need to understand that they are not engaging with a new system isolated from mathematics. On the contrary, they need to be aware that they are using these materials to support their understanding of the symbolic - mathematical system that they are studying.

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11 It is important to note that in 1950, Caleb Gattegno founded the *Commission internationale pour l'étude et l'amélioration de l'enseignement des mathématiques* (CIEAEM), of which he served as secretary until 1960. Among the members of CIEAEM were Jean Piaget, Jean-Louis Nicolet, Emma Castelnuovo, Trevor J. Fletcher, Gustave Choquet, and Jean Dieudonné, one of the most influential French mathematicians of the twentieth century, especially known for his association with the famous Bourbaki group (Powell, 2007). Additionally, in 1952 in England, Gattegno was one of the founders of the *Association for Teaching Aids in Mathematics* (which later became the *Association of Teachers of Mathematics*) and acted as the first editor of its journal, *Mathematics Teachings* (Powell, 2007). In 1953, CIEAEM requested Gattegno to evaluate the value of Émile-Georges Cuisenaire's materials. Cuisenaire had achieved something rare: his students enjoyed his work and understood it. With their arithmetic prowess aligned with the rigorous European curriculum, his students surprised educators (Trivett, 1959). In 1952, Cuisenaire published his work in a booklet entitled *Les Nombres en Couleurs*. However, for about 23 years, his work and inventions remained mostly unknown outside his village of Thuin, Belgium. In contrast, one year after the publication of his book, a providential meeting between this teacher and Gattegno resulted in the use of his materials in classrooms all over the world (Powell, 2007).

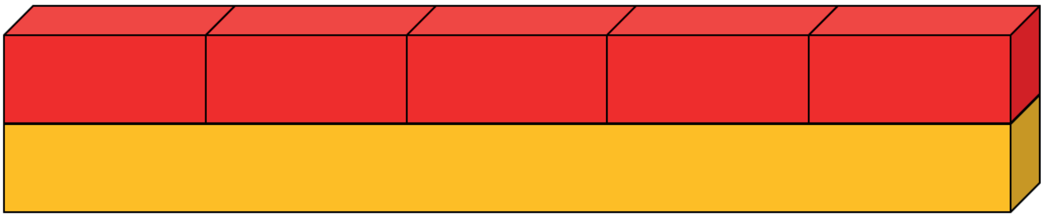
12 Base-ten blocks predate Dienes. For example, Zbigniew Antony Lubienski, also known as Roland A. Lubienski Wentworth, created and worked with base-ten blocks in the 1930s.

13 Also known as golden beads.

Still considering the possible limitations of using manipulatives, Thompson and Lambdin (1994) state that it is not easy to use them effectively. On the contrary, it is much easier to employ them inappropriately. Kilpatrick *et al*. (2001) highlight that manipulatives are not an end in themselves. For pedagogical purposes, they enable students to build meanings and connect different mathematical ideas.

It is important to consider the following distinction: although manipulative materials are designed or borrowed to explicitly and concretely represent abstract mathematical ideas (Moyer-Packenham, 2001), manipulatives do not embody mathematical meanings; in other words, mathematics is not inherent to the objects. Meira (1998) argues for what he terms as the *transparency* of manipulatives as an index of access to mathematical knowledge and activities - rather than a factor inherent to objects - and that unfolds through a process mediated by the use of the materials in a specific sociocultural practice. For example, in the classroom, the length of a red Cuisenaire rod may represent for students 1/5 of the length of an orange bar (Figure 1) only if they have previously developed mathematical activities with the rods and understood the comparison between the lengths of five red rods and one orange rod. On the contrary, the rod is simply a wooden or plastic object whose color, red or orange, does not convey any specific mathematical meaning to anyone.

Figure 1 - Comparison between the lengths of the red and the orange Cuisenaire rods.



Source: authors

Uttal *et al*. (1997, p. 50) maintain that the use of manipulatives can even be positive, but the objects do not embody any key to “unblock the mysteries of mathematics.” Other researchers also adhere to the notion that the manipulatives do not carry in themselves the meaning of the mathematical idea (Clements; Mcmillen, 1996). In this sense, Moyer-Packenham (2001) affirms that these materials do not inherently bear any type of meaning. It is the students’ reflections on their actions with the manipulatives that may support them in the construction of meanings. They are the ones who build meaning for the manipulatives, acting upon them through manipulation within a specific sociocultural practice.

MISTAKES REGARDING THE PRODUCTION AND USE OF MANIPULATIVES

In an attempt to map possible problems regarding the work with manipulatives, Silveira (2001) proposes the creation of three categories: manipulatives that were incorrectly built or designed, manipulatives that are used inappropriately, and manipulatives that are used for inadequate purposes.

VIDYA, v. 45, n. 1, p. 135-152,
jan./jun. - Santa Maria, 2025.

MANIPULATIVES THAT WERE INCORRECTLY BUILT OR DESIGNED

This category includes those manipulatives that, thanks to the inclusion of irrelevant features, careless manufacturing, or inappropriate labeling, cannot fulfill their function. Aligned with this perspective, Kaminski and Sloutsky (2013) state that those who design educational materials should limit the inclusion of "extraneous perceptual information" in their creations, avoiding that children have their attention diverted from the concept to be taught. Manipulatives "stripped off perceptual characters or irrelevant attributes" help children to concentrate their attention to the mathematical concept presented (Laski, *et al*, 2015).

To exemplify, Silveira (2021) presents the case of the colored abaci, as shown in Figure 2 below.

Figure 2 - Peg abacus or open abacus[14].

Source: Internet: standard abacus model commonly found in educational toy stores.

It is almost impossible to find, in physical or online stores, a peg abacus in which all the rings or beads are the same color. As a rule, on abaci, all the rings should represent a single unit. When placed in a different position, they begin to represent the powers of the given base specific to that position - in the case at hand, base 10. When the rings are colored differently, it opens up the possibility of interpreting the values of different powers of 10 as being associated with different colors, as a type of convention. It is not difficult to conclude that, even if it is not what the teacher intends, the beige rings in Figure 2 are associated with the value of one hundred units each, regardless of whether they are placed in the hundreds position on the abacus. In reality, the designer of this material thinks that the ring is worth one unit, but the idea of a unit is associated with the green rings. In this case, the abacus loses its use and function, because it would be sufficient to stack the corresponding number of rings, without using an abacus to obtain the desired number (Bartolini Bussi, 2011; Silveira, 2014; Silveira, 2016). For example, if there is a pile of rings containing one blue ring, two beige rings, and five green rings, knowing which abacus rods each color is used on, it is possible to deduce that the represented quantity is 1205. To eliminate the possibility of misinterpretation and ensure the material fulfills its intended purpose, the abaci must have all rings of a single color, regardless of the color chosen.

14 (Translation) Ten thousands, Thousands, Hundreds, Tens, Ones.

In a different setting, still in the category of manipulatives constructed or designed incorrectly, we point to the case of the Logical Blocks. Presented by Dienes and Golding (1966)[15], they aimed to help students develop elementary mathematical structures, such as the understanding set theory.

Figure 3 - Dienes logical blocks.

Source: authors' personal archive.

When presenting logical blocks or "attribute blocks," Dienes and Golding (1966) discuss the influence of the work of Vygotsky and William Hull on their creation and state that the latter author was the first to make a practical contribution to the use of blocks in the development of high-level logical thinking in children as young as 5. Among the conditions imposed by Hull for that to happen, there was the consideration that great care should be taken to ensure that *excessive verbalism* would not hinder concept formation (Dienes; Golding, 1966). The blocks, 3D geometric shapes, possibly as a "simplification" in relation to an "excessive verbalism", were named by Dienes and Golding (1966) according to the shapes of their largest faces or their bases. Thus, they were referred to as squares, oblongs, triangles, and circles (Dienes; Golding, 1966).

Certainly, Dienes and Golding (1966) were aware that this nomenclature did not correspond to the being blocks used, especially since it concerns very elementary mathematical knowledge. It seems that this nomenclature is used, in fact, under the illusion that the child will understand that, when requested to pick up a small, thin red circle, the teacher is only referring to the shape of the base of the small, thin red cylinder. In this case, it is likely that all the children will pick up the "correct" object, and it is possible that some may understand the name of the object refers only to the geometric shape of its base. However, it is likely that others will understand that that object is, in fact, a circle. They may come to understand that there are thin circles and thick circles, for instance. In this case, we may be facing two problems: the first is that the child might understand that these objects (and not their bases) carry these names. Thus, calling a cylinder, for instance, a circle. The second is that a circle already refers to another mathematical object.

15 In this text, we do not aim to discuss the influences of William Hull or Vygotsky on Dienes's creation, nor his "dispute" with Maria Montessori.

VIDYA

VIDYA, v. 45, n. 1, p. 135-152,
jan./jun. - Santa Maria, 2025.

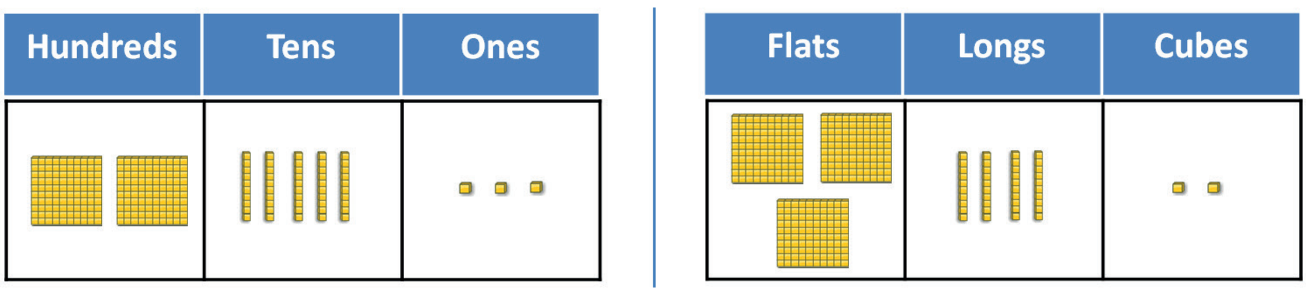
Possibly because this material can be used in the teaching and learning of mathematics since early childhood education, there is a preference for naming the blocks in a more simplified way. It would not be effective, for activities with very young children, to call a block a "triangular-based prism," which may have led the designer and their followers to name the block simply as "triangle." When working with children in early childhood education, using more general names that evoke the idea of three-dimensionality could help resolve this confusion. The blocks could be called round block, rectangular block, triangular block, and square block. In the early elementary years, the correct nomenclature for each block should be taught and used.

Something similar happens with Tangram puzzles, in which the pieces, which are 3D blocks, are referred to by two-dimensional figure names. It is commonly said that the Tangram is composed of two large triangles, a medium triangle, two small triangles, one square, and one parallelogram. The solution to this issue involves the same type of solution suggested for the case of logical blocks.

MANIPULATIVES USED INNAPROPRIETLY

This category deals with the cases in which the materials are well constructed and free from irrelevant characteristics, that is, they are adequate to be used in mathematics classrooms, but even so, are used inappropriately. A good example is the use of materials with a value other than one, whether apparent or conventional, placed over place value charts when teaching the decimal number system and its four basic operations (Silveira, 2014, 2016, 2018, 2019, 2020, 2021), such as presented in Figure 4.

Figure 4 - Base ten blocks placed over a place value chart or over labeled charts.



Source: Authors.

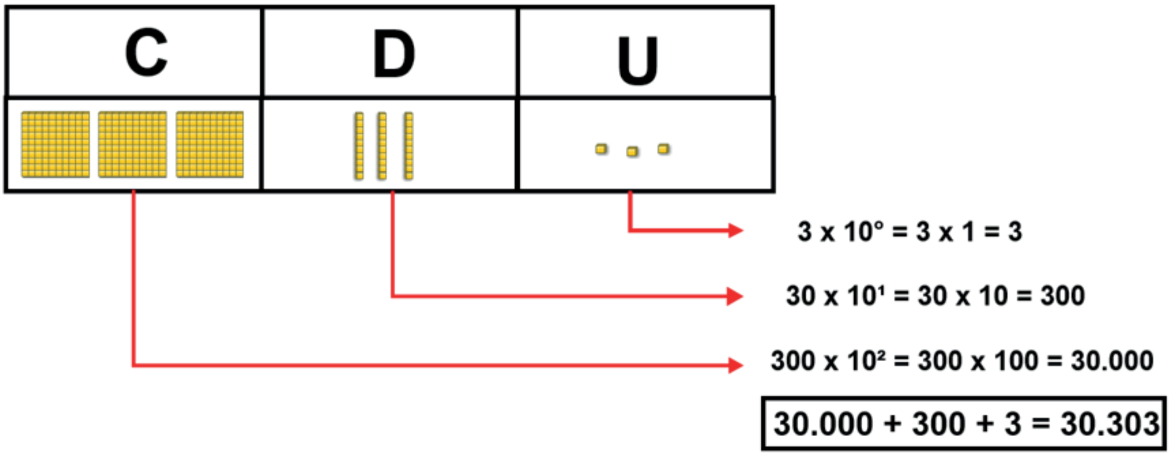
Silveira (2021) argues that "Base-ten blocks differ from abaci in that each type of block carries, embedded in its form, a value that translates a power of ten. Unit cubes are worth the value of units, longs are worth the value of tens, flats are worth the value of hundreds, and the cube represents a thousand" (p. 06), 10^0, 10^1, 10^2, and 10^3, respectively. The place value charts are structured on additive and multiplicative principles of our number system. The author states that a conflict arises when the blocks are introduced inside a place value chart.

> If a block that is worth 100 units (flat) is inserted in the order of the hundreds in a place value chart, what is supposed to happen? One may either consider that its absolute value is now 1 unit - which is absolutely incoherent, since students are constantly reminded that flats are worth 100 units - or its relative value becomes

> 10.000 units, for 100 (the added value of the flat) multiplied by 100 (multiplicative factor of the order of hundreds) equals 10.000. (Silveira, 2021, p. 07).

Figure 5 may offer a better understanding.

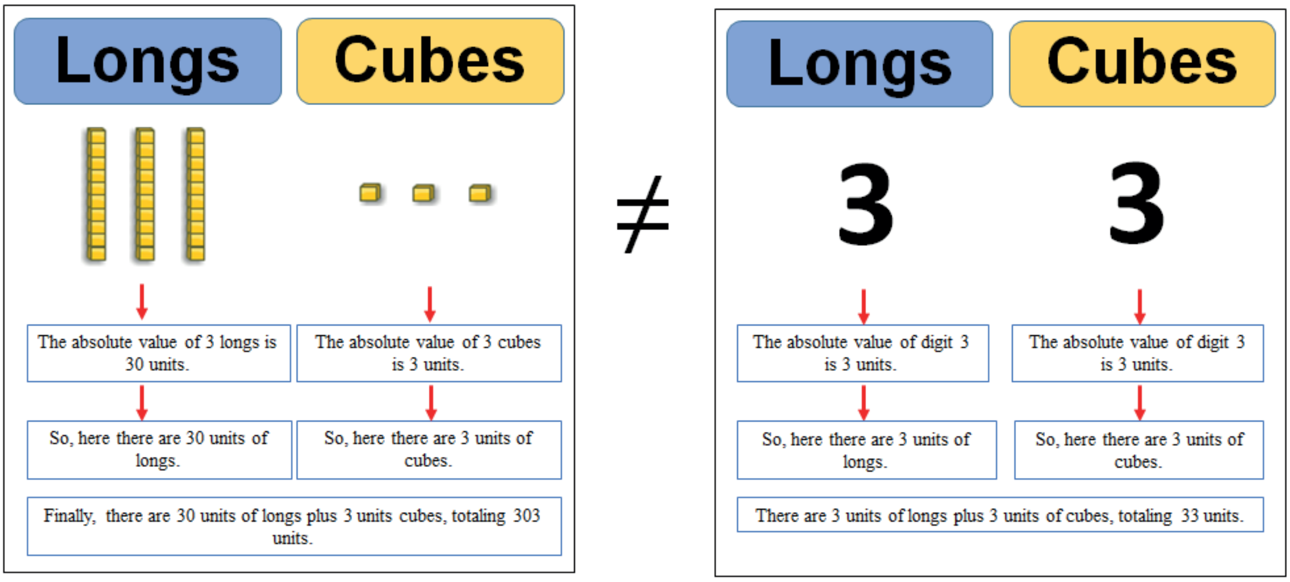
Figure 5 - Base-ten blocks placed over a place value chart and the possible consequences.



Source: Silveira, 2021, p. 07.

In Silveira (2021), one can also find an explanation for the difficulties that may arise in cases where the blocks are inserted in labeled charts. Figure 6 illustrates the discussion.

Figure 6 - Base-ten blocks placed over a labeled chart and the possible consequences.



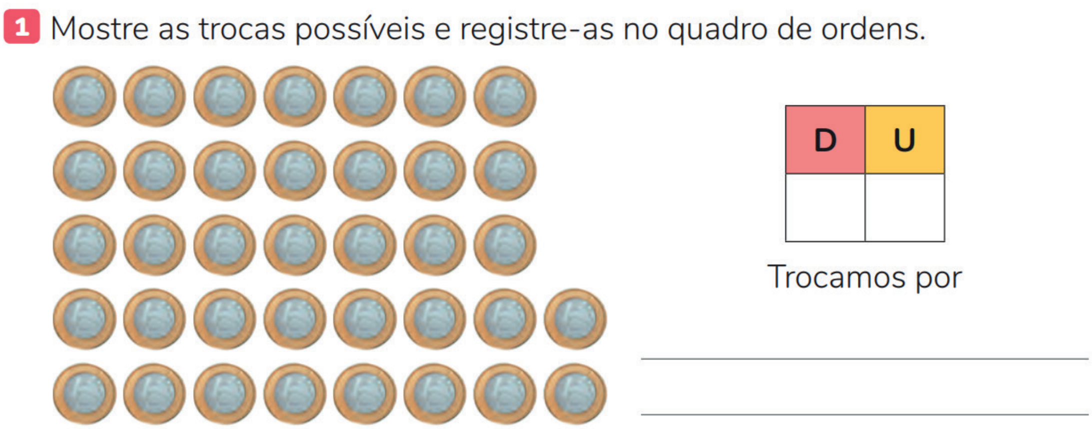
Source: Silveira, 2021, p. 08.

There are similar cases in which, among other base-ten materials, bundles of sticks or green chips[16] are used instead of base-ten blocks. All of these materials, including base-ten blocks, should not be inserted in place value charts, because they have aggregated values different from one unit.

MANIPULATIVES USED FOR INADEQUATE FUNCTIONS

In the third category, Silveira (2021) addresses the use of manipulatives for inadequate functions. When manipulatives serve inadequate functions, it means that a manipulative that is adequate to support the teaching of a certain mathematical content is used to teach other contents, creating a space for the emergence of comprehension problems. In the case of repeated use, a manipulative that had already been used to teach a specific mathematical content is reused to teach a different one, forcing the child to reinterpret the representation of the material. To offer an example of manipulatives used inappropriate functions, Silveira (2021) mentions the use of play money to teach the concept of base ten, such as presented in Figure 7:

Figure 7 - Play money being used to teach groupings of ten.[17]



Source: Rubinstein *et al.*, 2021, p. 26.

When the objective is to teach base ten for students, only materials that allow for the grouping of ten units of a given magnitude are useful, so that these can be exchanged for one unit of the next higher magnitude. In the case of play money, it only makes sense for students to exchange ten R$1.00 coins for a R$10.00 bill, or ten R$10.00 bills for a R$100.00 bill, or vice versa. However, in the Brazilian monetary system, which is the model for the play money bills, there are also R$2.00, R$5.00, R$20.00, R$50.00, and R$200.00 bills. The insert in the book previously mentioned includes bills for these values, except for the R$200.00 bill, so students can cut them out and manipulate them.

Given that all of these bills are known and even used on a daily basis by the students, problems may arise if a child decides to exchange two coins of R$1.00 for a R$2.00 bill, or five coins of R$1.00 for a R$5.00 bill. While R$2.00 and R$5.00 bills are part of the monetary system and are available

16 Material exclusively presented in the Ápis/Matemática collection, published by Ática Press, for the Early Years.
17 (Translation) Show the possible exchanges and record them in the place-value chart. We exchange for.

VIDYA, v. 45, n. 1, p. 135-152,
jan./jun. - Santa Maria, 2025.

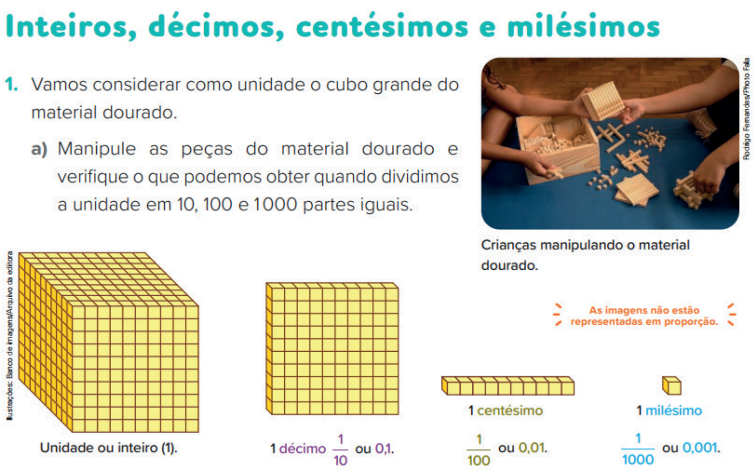
in the book insert, they are not appropriate for understanding the decimal system, because only the R$1.00, R$10.00, and R$100.00 bills represent powers of ten. A R$5.00 bill, for example, undermines the relationship established, as R$5.00 in this activity will always be represented by five R$1.00 coins. In base-ten blocks, for example, a material similar to play money, differing only in that it has apparent and non-conventional values, there is no block for five units. Thus, although play money is a useful material to teach basic arithmetic, it will only be efficient for teaching the base-ten system if the teacher introduces a fictional currency consisting only of R$1.00, R$10.00, R$100.00, and possibly R$1,000.00 bills.

Cases in which the use of a manipulative is extended to teach a new mathematical content relate to the notion of multivalent materials, as proposed by Gattegno (1958) and Puig Adam (1956-1957). According to our understanding, it is important to tackle the concept of manipulatives that have multiple uses and can be employed to represent many mathematical objects and, through their manipulation, carefully exemplify the multiple relations between these objects.

We understand, for example, that Cuisenaire rods can be used to represent multiple mathematical objects, such as natural numbers, integers and rational numbers (fractions and decimals), and the manipulation of this material can simulate various arithmetic operations, such as addition, subtraction, multiplication, division, exponentiation, logarithms, permutations, and combinations, as well as algebraic properties and formulas (Cuisenaire & Gattegno, 1954; Gattegno & Hoffman, 1976). However, care must be taken to create a safe environment for questions and investigation in the classroom. Finding and managing multiple mathematically and pedagogically legitimate uses for the same manipulative is not a trivial feat.

To exemplify, let us look at a case in which base-ten blocks are used to teach rational numbers in decimal representation. In these cases, either the cube (thousands unit) is taken as a unit, considering a flat (hundreds) as a tenth, a long (ten) as a hundredth, and a unit cube as a thousandth, or the flat (hundreds) is taken as a whole, working with longs and unit cubes as tenths and hundredths. Figure 8 illustrates this idea present in some textbooks.

Figure 8 - Base ten blocks being used to teach decimal numbers.[18]



Source: Dante & Viana, 2021, p. 172.

18 (Translation) 1. Let's consider how a large cube of the base-ten material. a) Manipulate the base-ten material e verify what we can obtain when we divide a unit in 10, 100, and 1000 equal parts.
Unit or whole; 1 tenth, 1/10, or 0.1; 1 hundredth, 1/100, or 0.01.
Children manipulating the base-ten material.
The imagines are proportional.

Usually, textbooks themselves introduce base-ten blocks in the first or second grade. In these cases, the goal is for the material to support the teaching and learning of the number system and the four basic arithmetic operations. Nevertheless, symbolically reinterpreting a material is not so simple. According to DeLoache (2000), "[...] to use a symbolic object as a model, map, or image, one must achieve dual representation; that is, one must mentally represent both the symbol itself and its relation to its referent" (p. 329). In this case, the "symbol," which refers to the base-ten blocks, besides being physical objects representing themselves (wooden blocks), are used during the early elementary years to represent units, tens, hundreds, and thousands. When they are used again later, the student must be able to again ascribe meaning to (re-signify) the "symbol," which from that moment on becomes a triple representation, representing itself; the first four orders of whole numbers and the unit; and the three largest orders of numbers less than one. For this task to be successful, the child must replace one referent with another. This may be accessible for some children but not for others. In such cases, the ideal would be to use a manipulative that has not yet been employed for other purposes.

To some extent, the inappropriate use of this category of manipulatives depends on the pedagogical work developed in the classroom, by the teacher, or even on how manuals and textbooks suggest that manipulatives should be used. We can speak of a mathematics classroom culture, from the perspective of problematization, in which students are accustomed to manipulating materials in different mathematical contexts. For example, a practice using Cuisenaire rods in a mathematics class in which the students experiment with assigning and reassigning the rods according to their value, by varying which one represents the unit, while the remaining rods take on variable values, fostering a culture of assigning and reassigning meaning to the materials. In this process, the limitations and potentialities of each material also become variable. A student who is used to such reinterpretations will likely have no trouble understanding that, in the base ten blocks, the cube, which in the decimal number system represents 1000, can represent 1, and thus the flat represents 1/10, the long represents 1/100, and the unit cube, 1/1000.

Uttal (2003) affirms that the role of the teacher is key to establish whether the manipulatives will aid, hinder, or make no difference in children's understanding of mathematics. According to him, it is the teacher's role to conduct the orientation in an effective way. Even if all the steps are properly executed, when manipulatives are used in a routine basis, children's learning may not be enhanced (Clements & Mcmillen, 1996).

Therefore, for an effective support of mathematical learning, it is necessary to be aware of the constructions and characteristics of the manipulatives in use, as well as of their limits, potentialities, and pedagogical uses within the cultural context of the classroom.

FINAL REMARKS

Our goal in this text was to provide an understanding of manipulatives in Mathematical Education, highlighting their uses, as well as necessary precautions and common misconceptions in their construction and use in pedagogical practices. The concerns were focused on both the construction and on how they can be appropriated/adapted, re-signified, and used in the classroom.

Following other authors, in this text we have constructed and presented a proposal for categorizing the most diverse types of manipulatives in mathematical education. These categories comprise didactically constructed materials, culturally inherited instruments inherited from tradition and objects taken out of day to day life. In addition to physical objects, they also include pictorial objects - that can be cut out from books, as well as inserts and other resources that the student can manipulate - and virtual objects, understood as visual interactive representations of dynamic mathematical objects enabled by computers.

Manipulatives are important in mathematics education, but their use is not a guarantee of success. Their efficiency and effectiveness appear to be related to three variables: the choice of material, clear and participatory teacher instruction and student's participation through a mathematical process. Regarding material choice, the text addresses misconceptions related to they production and use, namely: manipulatives that were incorrectly built or designed, manipulatives inappropriately, and materials used for inadequate functions. This text is connected to other ongoing studies (Silveira, 2014, 2016, 2018, 2019, 2021) that highlights a series of problems regarding the misuse of manipulatives with students. We believe it is essential that teachers and researchers consider these discussions in their researches and practice in Mathematics Education.

We emphasize that, for any intended use with a material, the most important moment is perhaps that students experience manipulating them freely, becoming familiar with their characteristics and properties, so they can make a good use of them when challenged by the problems proposed by the teachers or by the textbook authors.

We believe this text presents a possible categorization for manipulatives, their uses and misuses, opening up possibilities for other categorizations as well as of their expansion. Further research could address teachers' understanding regarding manipulatives and their use in pedagogical practices as well as the impacts of misuse or misapplications of these resources on students' mathematical learning processes.

REFERENCES

Baroody, A. J. (1989, October). Manipulatives don't come with guarantees. **National Council of Teachers of Mathematics**, 37(2), 4-5.

Bartolini Bussi, M. G. (2011). Artefacts and utilization schemes in mathematics teacher education: place value in early childhood education. **Journal of Mathematics Teacher Education**, 14(2), 93-112.

Bartolini Bussi, M. G., & Boni, M. (2003, July). Instruments for semiotic mediation in primary school classrooms. **For the Learning of Mathematics**, 23(2), 15-22.

Brown, M. C., Mcneil, N. M., & Glenberg, A. M. (2009). Using concreteness in education: Real problems, potential solutions. **Child Development Perspectives**, 3(3), 160-164.

Clements, D. H., & Mcmillen, S. (Jan. 1996). Rethinking "concrete" manipulatives. **National Council of Teachers of Mathematics**, 2(5), 270-279.

Cuisenaire, G., & Gattegno, C. (1954). **Numbers in colour: A new method of teaching the process of arithmetic to all level of the Primary School**. Hienemanns.

Dante, L. R., Viana, F. (2021). **Ápis Mais: Matemática, 5º ano** (1th ed.). Editora Ática S. A.

Deloache, J. S. (2000). Dual representation and young children's use of scale models. **Child Development**, 71(2), 329-338.

Dienes, Z., & Golding, E. (1966). **Learning logic, logical games**. New York: Herder & Herder.

Furner, J. M., Yahya, N., & Duffy, M. L. (2005, September). Teach mathematics: strategies to reach all students. **Intervention in School and Clinic**, 41(1), 16-23.

Gattegno, C. (1958). Les matériels multivalents. In: C. Gattegno et al. (1958), **Le matériel pour l'enseignement des Mathématiques** (pp.191-201). Delachaux et Niestlé.

Gattegno, C. (1958/2009). **Geometry**. Educational Solutions Worldwide.

Gattegno, G. C., & Hoffman, M. R. (1976). **Handbook of activities for the teaching of mathematics at the elementary school**. Human Education.

Golafshani, N. (2013). Teachers' beliefs and teaching mathematics with manipulatives. **Canadian Journal of Education/ Revue Canadienne de L'Éducation**, 36(3), 137-159.

Kaminski, J. A., & Sloutsky, V. M. (2013). Extraneous perceptual information interferes with children's acquisition of mathematical knowledge. **Journal of Educational Psychology**, 105(2), 351-363.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). **Adding it up: Helping children learn mathematics**. National Academy Press.

Ladel, S., & Kortenkamp, U. (2016). Artifact-centric activity theory - a framework for the analysis of the design and use of virtual manipulatives. In: P. Moyer-Packenham (org.), **International perspectives on teaching and learning mathematics with virtual manipulatives**. Mathematics education in the digital era. Springer International Publishing, 7, 25-40.

Laski, E. V., Jor'dan, J. R., Daoust, C., & Murray, A. K. (2015). What Makes Mathematics Manipulatives Effective? Lessons From Cognitive Science and Montessori Education. **SAGE Open**, 5(2), 1-8.

Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in Mathematical activity. **Journal for Research in Mathematics Education**, 29(2), 121-142. https://doi.org/10.2307/749895

Moyer-Packenham, P. S., & Bolyard, J. J. (2016). Revisiting the Definition of a Virtual Manipulative. In: Moyer-Packenham, P. (eds) **International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives**. Mathematics Education in the Digital Era, vol. 7. Springer, Cham. https://doi.org/10.1007/978-3-319-32718-1_1

Moyer-Packenham, P. S. (2001). Are we having fun yet? How teachers use manipulatives to teach Mathematics. **Educational Studies in Mathematics**, 47(2), 175-197. www.jstor.org/stable/3483327

Moyer-Packenham, P. S., & Jones, M. G. (2004). Controlling choice: teachers, students, and manipulatives in mathematics classrooms. **School Science and Mathematics**, 104(1), 16-31.

Powell, A. B. (2007). Caleb Gattegno (1911-1988): A famous mathematics educator from Africa? **Revista Brasileira de História da Matemática**, número especial Festschrift Ubiratan D'Ambrosio, 199-209. https://doi.org/10.47976/RBHM2007vn17.

Puig Adam, P. (1956-1957). Les mathématiques et le concret [Mathematics and the concrete]. **Mathematica & Paedagogia**, (12), 62-65.

Rubinstein, C., França, E., Ogliari, E., Miguel, V., & Resende, E. (2021). **Bem Me Quer - Mais Matemática, 3º ano** (1th ed.). Editora do Brasil.

Silveira, E. (2014). Afinal, estamos ensinando matemática errado? In: **Anais do 10ª ANPED SUL**, Florianópolis-SC, Brasil: Universidade do Estado de Santa Catarina.

Silveira, E. (2016). Materiais manipuláveis e alguns riscos que envolvem sua utilização. In: Silveira, E. et al. (orgs.). **Alfabetização na perspectiva do letramento: letras e números nas práticas sociais** (vol. 1, pp. 221-240). NUP-CED-UFSC.

Silveira, E. (2018). Afinal, está certo ou errado? Um estudo sobre indicações de uso de blocos base dez em livros didáticos de matemática no Brasil. In: **Anais do 7º Seminário Internacional de Educação Matemática**. Foz do Iguaçu-PR, Brasil: SBEM. http://www.sbemparana.com.br/eventos/index.php/SIPEM/VII_SIPEM/schedConf/presentations

Silveira, E. (2019). Materiais manipuláveis e alguns riscos que envolvem sua utilização. In: **Anais do 13º Encontro Nacional de Educação Matemática**. Cuiabá-MT, Brasil: SBEM.

Silveira, Everaldo (2021). A Study on the indications to the use of base ten blocks and green chips in mathematics textbooks in Brazil. **The Mathematics Enthusiast**, (18) 3, 469-501. DOI: https://doi.org/10.54870/1551-3440.1534

Silveira, E. (2021). Materiais manipulativos: apresentando três chances de dar errado. In: **Anais do 8º Encontro Catarinense de Educação Matemática**. Regional Santa Catarina (evento virtual), Brasil: SBEM/SC.

Thompson, P. W., & Lambdin, D. (1994). Concrete materials and teaching for mathematical understanding. **The Arithmetic Teacher**, 41(9), 556-558. http://www.jstor.org/stable/41196106

Trivett, J. V. (1959). The coloured sticks: A teacher's description of the Cuisenaire-Gattegno approach to the teaching of mathematics. **The New Scientist**, 6(160), 1183-1186.

Uribe-Flórez, L., & Wilkins, J. (2010). Elementary school teachers' manipulative use. **School Science and Mathematics**, 110(7), 363-371.

Uttal, D. H. (2003). On the relation between play and symbolic thought: The case of mathematics manipulatives. In: Saracho, O. N., & Spodek, B. (orgs.), **Contemporary perspectives on play in early childhood education** (pp. 97-114). Information Age Publishing.

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997, January 01). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. **Journal of Applied Developmental Psychology**, 18(1), 37-54.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2013). **Elementary and Middle School Mathematics: Teaching Developmentally** (8th ed.). Pearson.