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THE DEVELOPMENT OF MATHEMATICAL THINKING THROUGH ADAPTIVE EDUCATION: THE CASE OF A LINEAR ALGEBRA COURSE

O DESENVOLVIMENTO DO PENSAMENTO MATEMÁTICO ATRAVÉS DA EDUCAÇÃO ADAPTATIVA: O CASO DE UM CURSO DE ÁLGEBRA LINEAR

EL DESARROLLO DEL PENSAMIENTO MATEMÁTICO MEDIANTE LA EDUCACIÓN ADAPTATIVA: EL CASO DE UN CURSO DE ÁLGEBRA LINEAL

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ABSTRACT

The one-size-fits-all model is widely used in teaching and learning processes around the world. This model does not recognize students' learning conditions and preferences. As an alternative, the adaptive education model has been developing in recent years. In mathematics, the current state of this adaptive education model is oriented to the learning of objects and algorithms and not to the development of mathematical thinking. To overcome this shortcoming, the design-based research approach was used to create and implement an adaptive learning management system framed in a module of a linear algebra course, in which adaptation was done based on modes of thinking (SIERPINSKA, 2000). The results of this implementation show that people manifest diverse modes of thinking when conceiving mathematical objects and that these modes affect the way they learn. These results also highlight some existing limitations for adaptive educational processes such as these to generate greater impacts.

Keywords: Adaptive Learning Management System. Mathematical thinking. Linear algebra (vectors). Adaptive education. Modes of thinking.

RESUMO

O modelo de tamanho único é amplamente utilizado nos processos de ensino e aprendizagem no mundo. Este modelo não reconhece as condições e preferências de aprendizagem dos estudantes. Como alternativa, o modelo de educação adaptativa tem sido desenvolvido nos últimos anos. Em matemática, o estado atual deste modelo de educação é orientado para a aprendizagem de objetos e algoritmos e não ao desenvolvimento do pensamento matemático. Para superar esta deficiência, a abordagem de pesquisa baseada no desenho foi utilizada para criar e implementar um sistema de gestão de aprendizagem adaptativa enquadrado num módulo do curso de álgebra linear, no qual a adaptação foi feita em função dos modos de pensamento (SIERPINSKA, 2000). Os resultados desta implementação mostram que as pessoas manifestam diferentes modos de pensar quando concebem objetos matemáticos e que estes modos afetam a maneira como aprendem. Estes resultados também mostram algumas limitações existentes para que processos educacionais adaptativos como estes gerem maiores resultados.

Palavras-chave: *Sistema de Gestão de Aprendizagem Adaptativa. Pensamento matemático. Álgebra linear (vetores). Educação adaptativa. Modos de pensar.*

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RESUMEN

El modelo de talla única se utiliza ampliamente en los procesos de enseñanza y aprendizaje en todo el mundo. Este modelo no reconoce las condiciones y preferencias de aprendizaje de los estudiantes. Como alternativa, el modelo de educación adaptativa se ha ido desarrollando en los últimos años. En matemáticas, el estado actual de este modelo de educación adaptativa está orientado al aprendizaje de objetos y algoritmos y no al desarrollo del pensamiento matemático. Para superar esta carencia, se utilizó el enfoque de la investigación basada en el diseño para crear e implementar un sistema de gestión del aprendizaje adaptativo enmarcado en un módulo de un curso de álgebra lineal, en el que la adaptación se realizó en función de los modos de pensamiento (SIERPINSKA, 2000). Los resultados de esta implementación muestran que las personas manifiestan diversos modos de pensamiento al concebir los objetos matemáticos y que estos modos afectan a la forma de aprender. Estos resultados también ponen de manifiesto algunas limitaciones existentes para que procesos educativos adaptativos como estos generen mayores impactos.

Palabras clave: *Sistema de gestión del aprendizaje adaptativo. Pensamiento matemático. Álgebra lineal (vectores). Educación adaptativa. Modos de pensamiento.*

INTRODUCTION

Throughout much of human history, education was a personalized phenomenon. Masters taught their arts or crafts to a very small, often unitary number of apprentices. This did not necessarily imply that teachers adapted their ways of teaching to the learning conditions of their apprentices, but it did generate a tendency for this to happen.

Then, processes of massification of education took place. In fact, these processes, rather than being characterized by massification, were characterized by a uniformization of education. In the middle of the 18th century, because of the First Industrial Revolution, it became necessary to train many people, laborers, to work in industries doing repetitive tasks, which implied that they were trained in routine and obedience. The massification of education gave rise to the one-size-fits-all model, which was characterized, as mentioned above, by the standardization of educational processes, which led to disregard for the differential learning conditions of students and the teacher's way of thinking took precedence.

Based on this situation, at the end of the 20th century and the beginning of the 21st century, several authors (BETTAHI, 2018; BRUSILOVSKY, 2000; GRAF et al., 2012; POPESCU; BADICA; MORARET, 2010; POWELL; KUSUMA-POWELL, 2011; YARANDI; TAWIL; JAHANKHANI, 2012; etc.), both in the field of education and in the related field of educational technologies, began to propose solutions to overcome the one-size-fits-all model. These solutions are called by various names: personalized education or adaptive education and are characterized by generating curricular flexibility to cater in a discriminating manner to the greatest number of learners.

These solutions are basically of two types: intelligent tutoring systems, which adapt content to the learner but within certain boundaries; and adaptive hypermedia systems, which provide content and navigation paths that adapt to the user's needs (YARANDI; TAWIL; JAHANKHANI, 2012).

Both types of solutions are characterized by being mediated by technology, which is natural because no teacher in the world would be able to teach in a reasonable time and with a reasonable effort to a moderately large group of students attending to their diversity of learning conditions. The mediation of technology in education has also benefited from online learning resources, the availability of student data, and advances in artificial intelligence and machine learning techniques.

There are various learning conditions that are susceptible to be addressed in a pedagogical process. Powell et al. (2011 *apud* BETTAHI, 2018, p. 2) identifies different learning conditions, "such as ethnicity, culture, linguistic origin, socioeconomic status, religious faith, learning difficulties, gifted students, and attention problems, among other aspects". But, in addition to this diversity of conditions in students, advances in neuroscience have made it possible to generate other differentiating aspects, such as learning styles, ways of thinking and students' interests.

In summary, learning conditions can be classified into three dimensions (BETTAHI, 2018):

- Learning abilities and needs: Students show different degrees of strengths and weaknesses in learning, and their identification allows adapting the level of difficulty of the contents and learning experiences and the level of support they need.
- Interests and learning goals: Based on the characterization of the students in this dimension, differentiated routes can be generated according to their interests and curiosity, and making them establish their own learning goals. Addressing this dimension implies the formulation of individualized study plans with different learning paths.
- Learning preferences: This dimension includes learning conditions such as the students' language, the learning medium, the type of information input and the way it is processed.

Specifically in the third dimension, one of the most studied learning conditions is learning styles, a characteristic that refers to the different ways in which people receive and process information, which constitutes a tendency and not a determinant, because even if a person has certain preferences, it does not mean that there is only one way in which he or she can learn.

There is a great amount of research related to learning styles in general and their use for learning based on technological solutions developed models of learning styles (DUNN; DUNN; PRICE, 1981; FELDER; SILVERMAN, 1998; HONEY; MUMFORD, 1989; KEEFE, 1987; KOLB, 1985; MYERS; MCCAULLEY; MOST, 1985; etc.); and many other researchers have used this learning condition to develop technological systems for learning in various fields.

However, despite this wide variety, there are also several studies that claim that taking learning styles into account in educational processes does not have a positive effect (DEMBO; HOWARD, 2007; KIRSCHNER, 2017; ROSÉ et al, 2019; etc.).

In that sense, there is no strong evidence of the benefits of taking this learning condition into account, which depend exclusively on personal conditions, but its impact seems to depend on other additional factors.

Another characteristic of research based on the development of technological systems for education based on learning styles, specifically in the field of mathematics, is that they are oriented to the learning of mathematical objects, but not to the development of thinking, which may be due to the researchers involved in these studies having as their primary line of training fields of engineering and not mathematics or mathematics education.

Another learning condition, more related to mathematics, concerns modes of thinking. This learning condition was proposed by Sierpinska (2000) specifically for the learning of linear algebra and is the one considered for the present research.

In summary, this article reports the results of the implementation of an adaptive learning management system designed to consider modes of thinking in the teaching of a module on vectors in a linear algebra course, with the purpose of developing mathematical thinking. These results are manifested in terms of two factors: students' perceptions of their learning experience with the system and evidence of students' use of mathematical thinking when interacting with the system.

THEORETICAL FRAMEWORK

The design of an adaptive system involves the characterization of the three models that constitute it (MURRAY; PEREZ, 2015):

- The learner model is a representation of the learner making explicit his or her characteristics relevant to the adaptation, such as his or her personal information, cognitive traits, initial knowledge level with respect to the domain and particularities with respect to the learning conditions being considered for the adaptation. In addition, this model provides mechanisms to evaluate the learner's performance.
- The domain model is a representation of knowledge in a specific domain. The knowledge must be adapted to the learner's learning conditions and preferences and, for this reason, it is necessary to establish the learning units and the relationships between them, so that the possible routes that the learner will follow can be determined. These learning units must be arranged in the form of a repository of well-described resources in terms of the adaptation process (physical characteristics, knowledge characteristics, instructional function, and specification of relationships).
- The adaptation model defines how the adaptation will be performed based on the data it takes from the learner model and its relationship to the domain model. There are several ways of effecting the adaptation. It can be done at the beginning of the process, or it can be dynamic throughout the process, in stages or constantly. One can also think of adapting various curricular elements: the purposes (their orientation and depth), the contents (their order of presentation, their depth, their nature) or the assessment (its complexity).

The implementation of these three models takes the form of an adaptive learning management system (ALMS), i.e., a computational system that characterizes the learner and stores his or her parameters, in which the domain model is loaded, and which is, in effect, responsible for adaptively arranging the content for the learners.

These ALMS differ from traditional learning management systems (LMS), since the latter provide the same content to all learners in a linear fashion, i.e., without considering the learning conditions.

In particular, the learning conditions used in this research and described in the learner model are called modes of thinking and were proposed by Sierpinska (2000) specifically for learning linear algebra; however, it is possible to determine analogous modes of thinking for other mathematical content domains.

In her article "On some aspects of students' thinking in linear algebra" (SIERPINSKA, 2000, p. 209), the researcher proposes that

> three modes of reasoning in linear algebra will be distinguished, corresponding to its three interacting 'languages': the 'visual geometric' language, the 'arithmetic' language of vectors and matrices as lists and tables of numbers, and the 'structural' language of vector spaces and linear transformations.

She also indicates that although these three modes of thinking appeared chronologically in the history of mankind, none of them has replaced the others, but rather they coexist in the understanding that human beings have of linear algebra.

Nevertheless, it can be proved that each person has a preferred mode of thinking, especially between the first two: the synthetic-geometric and the analytic-arithmetic. The analytic-structural mode is configured as a mode of arrival, rather than a mode of departure. Or in other words:

> But the most interesting fact is that linear algebra can be seen as the result of an overcoming of two obstacles or two opposed dogmatic positions: one refusing the entry of numbers into geometry, and the other that of 'geometric intuition' into the pure domain of arithmetic (SIERPINSKA, 2000, p. 209)

Just as none of the modes of thinking has been replaced by the others in history, neither has this happened in the mind of the learner. Even if there is a preferred mode, people use different modes to think about the characteristics of mathematical objects.

A first differentiation between modes of thinking is in the synthetic versus analytic disjunction. In the synthetic (geometric) mode of thinking the mind describes objects that are given to it directly. For example, a circle is an object of which it has a prior notion, a notion of its shape, and from this can be derived properties that characterize but do not define it. On the other hand, in the analytical modes of thinking (arithmetical and structural) the objects are given indirectly, so for example, the same circle of the previous example is given by a series of conditions that define it, whether they are arithmetic or structural conditions.

As said before, this disjunction has a historical origin that Sierpinska (2000) describes:

> The development of linear algebra started as a process of thinking analytically about the geometric space. Taking a rather broad perspective, we could distinguish, in this development, two large steps related to two processes. One was the arithmetization of space, as it took place in the passage from the synthetic geometry to the analytic geometry in . The other was the de-arithmetization of space or its structuralization, whereby vectors lost the coordinates that anchored them to the domain of numbers and became abstract elements whose behavior is defined by a system of properties or axioms.

The second differentiation to be made is between the analytical modes of thinking: arithmetic and structural. In the analytic-arithmetic mode of thinking, objects are defined by formulas, which establish a mathematical relationship between their characteristics and allow them to be defined arithmetically, i.e., to be calculated unambiguously. On the other hand, in the analytical-structural mode of thinking, an object is defined by a set of properties, i.e., the calculation takes a back seat and what matters now is the enunciation and verification of its properties.

Below is a table showing the different ways of thinking about some of the mathematical objects related to vectors.

Table 1 - Mathematical objects related to vectors in different modes of thinking.

Object	Synthetic-geometric	Analytic-aritmetic	Analytic-structural
Vectors	Arrows on the line, plane or in space, with coordinates	'Boxes' with organized numbers	Do not look at the components inside the vectors, look at the vectors, as a whole, as elements of a space
Operations with vectors	Movements, deformations and compositions of arrows	Operations with numbers in 'boxes'	Formally defined operations, with abstract properties
Norm	Arrow length	A number that is calculated with a certain algorithm (larger components give a larger number)	It is the operational definition of the (Euclidean) distance function. Any function that associates an element of the vector space with a real number, and that satisfies some properties
Scalar product	Position of one arrow with respect to the other arrow	An algorithm associating two vectors with a real number	A function that is defined from the Cartesian product of the space itself in the base field and satisfies properties that are independent of the space and independent of the field (contained in).

Source: Prepared by the authors

The adoption of modes of thinking as a learning condition for adaptation in the adaptive model developed for this research produces two important effects for students' education.

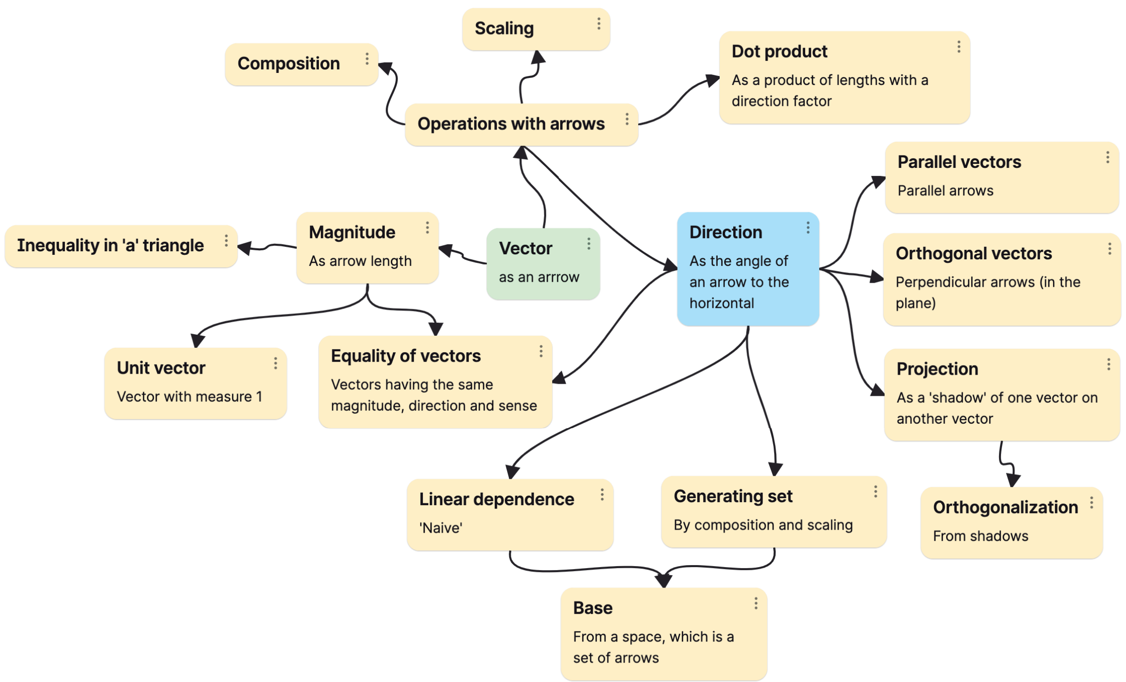
The first effect is that it is particularly coincident with the intention of this research to place a major emphasis on the development of mathematical thinking and not just on learning mathematics. That is, through the development and implementation of this adaptive model, the aim is to get students to think using mathematical objects and not just to learn to do calculations with them.

The key strategy for developing students' mathematical thinking is the resolution of challenging or non-routine problems. For students to be able to solve this type of problems, it is necessary to go beyond instrumental understanding to relational understanding of mathematical objects (SKEMP, 1976). Instrumental understanding describes the state in which the student knows rules without reasons, i.e., knows the "how" of mathematical objects, but does not know the "why". In this sense, instrumental understanding allows the use of specific tools in local contexts, while relational understanding is characterized by the construction of schemas from which plans can be devised to solve non-routine situations.

The second effect has to do with the sequencing order of the content in the domain model. In this adaptive system, not only are the students' trajectories of progress being adapted according to their initial mode of thinking, but also in each trajectory there are different ways of sequencing the contents. For the sequencing of contents, the concept of genetic decomposition, taken from APOS theory (DUBINSKY; LEWIN, 1986), was used.

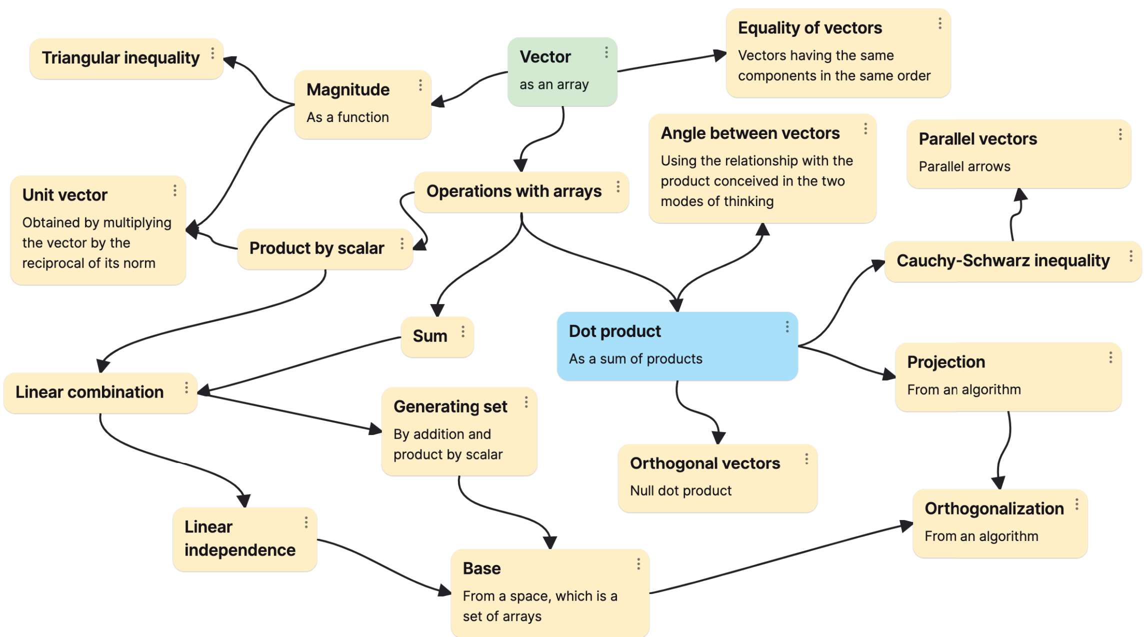
In this sense, for example, the following two figures show the difference between the order of mathematical objects (which were worked on in the module developed in this research) related to vectors for the synthetic-geometric mode of thinking and for the analytical-arithmetic mode of thinking.

Figure 1 - Genetic decomposition of content in the synthetic-geometric mode of thinking.



Source: Prepared by the authors

Figure 2 - Genetic decomposition of content in the analytic-arithmetic mode of thinking.



Source: Prepared by the authors

The first genetic decomposition is centered on the concept of vector, conceived as an arrow, so that the other concepts are related to that conception. The centrality of the concept of direction, which is not evident in the first case, is observed. The genetic decomposition shown in the second figure also focuses on the concept of vector, but in this one the dot product takes center stage, thus, the definition of the norm of a vector in the synthetic geometric mode is introduced as a geometric concept, but in the analytic arithmetic mode it is necessary to first understand the definition of the dot product between vectors to be able to use an arithmetic definition of the norm of a vector.

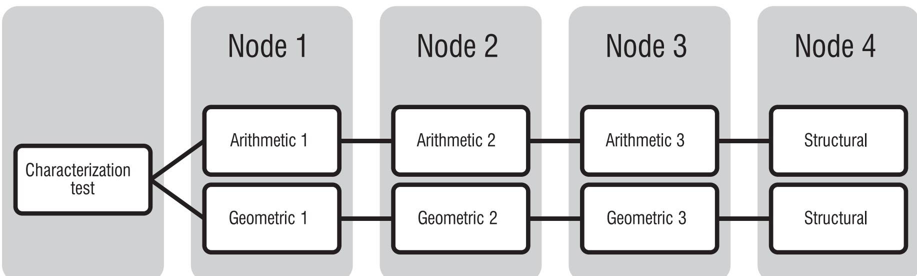
It has been mentioned that the concept of genetic decomposition is included in the APOS theory, so it is worthwhile to introduce a brief description of this theory in this conceptual framework. APOS theory was proposed by Ed Dubinsky in the year 1983 and has had a fruitful development, not only by this author, but by a strong stream of researchers in mathematics education, as can be seen, for example, in the book APOS Theory, A Framework for Research and Curriculum Development in Mathematics Education (ARNON, et al., 2014). This theory is based on the hypothesis that the development of mathematical thinking occurs in individuals when faced with mathematical problem situations by first constructing mental actions of a particular nature, then abstracting these actions into general processes, and then encapsulating these processes in mathematical objects, which will be organized, finally, into schemes that will allow them to make sense of the situations and solve not only the problem originally posed, but others in which the same scheme can be used.

METODOLOGY

The results reported in this article are the product of a study framed in the design-based research method. (BROWN, 1992). Following the guidelines of this research method, two iterations of implementation of an adaptive curriculum design based on the elements of the theoretical framework and deployed in an ALMS were conducted. The first iteration was carried out with students of a course on Linear Algebra at the Fundación Universitaria Konrad Lorenz, Bogotá, Colombia, during the semester 2021-1. This iteration was implemented by the two professors of the subject and directed by the researcher. The 42 students of the course were divided into three class groups. The second iteration was carried out with students of a course on Problem Solving at the Universidad Antonio Nariño, Bogotá, Colombia, during the semester 2021-2. This iteration was implemented by the researcher. The 70 students of the course were divided into two class groups. As in the first iteration, in this one the curriculum was developed for the module on vectors that was part of the course.

The ALMS had the following form:

Figure 3 - Structure of the adaptive system.



Source: Prepared by the authors

In the figure above it can be seen that students must first answer the characterization test on modes of thinking. In this test, students are classified according to whether they prefer the synthetic-geometric mode of thinking or the analytical-arithmetic mode of thinking. Based on their answers, they will be oriented to start at node 1 in one of the two adaptation lines. Node 1 corresponds to the first lesson, the topics of which will be defined from genetic decomposition. The topics of each of the following levels will be defined in the same way.

Each level of the nodes should allow students to understand the concepts and algorithms, do exercises involving them, and expand their scope and profundity by solving challenging problems. Each node is divided into the following stages or "moments".

- Moment that introduces the node: this includes the objective and contents of the node.
- Moments of theoretical development: the explanation of the theory required to solve the exercises and problems is presented here in text and video form.
- Moments with questions requiring comprehension: they contain questions to be solved by the students. After each of these moments, the students find the respective feedback for the questions, with which they will be able to verify their understanding of the theory presented. If they have doubts when verifying their answers, they can interact with (ask) the teacher.
- Moments with exercises: these are intended to help students become proficient when dealing with the algorithms presented alongside the theory. After each exercise moment, feedback is presented for students to check their results. If they have concerns about the exercises, they can ask the teacher.
- Moments with problems: students will be presented with challenging problems. The feedback on the solution to these problems is provided directly by the teacher to the students. For this purpose, in the teacher's platform there are texts with possible feedback, which help him/her to better guide the students.
- Moment with node map: here students will find a summary of what they have learned during the node.

This ALMS was built with the following characteristics:

- It was an adaptable and non-adaptive system, as proposed by Mudrák (2018).
- Adaptation was static in nature, i.e., students were classified at first and progress through the system was based on that classification.
- When the student accessed it for the first time after registration, he or she was presented with the test designed to characterize his/her mode of thinking; in subsequent interactions the student accessed the nodes directly.
- The system presented each ~~node~~ moment of the node in the form of a card and prompted the student to advance to the next moment when appropriate.
- When the student reached the end of each node, he or she had to use the system to request authorization from the teacher to continue to the next node.
- The system saved the student's progress in each work session.
- The system had three user profiles: administrator, teacher and student, and for each of them there were different interfaces.
- The teacher interface allowed grading the problems, recognizing the progress status of the students and storing the thinking mode in which each was classified.

The test to characterize a student's type of thinking was designed and validated to classify students into two groups: those who tended to think synthetically-geometrically and those who tended to think analytically-arithmetically. The text can be found in the appendix. Four factors were used to characterize each student's mode of thinking:

Factors associated with the synthetic-analytic disjunction include:

- Action or pretension: refers to the ways in which people act in situations.
- Attitude: refers to the posture that people manifest when faced with situations.
- Factors associated with geometric-arithmetic disjunction are as follows:
- Relation to objects: refers to the way people perceive objects to facilitate their understanding.
- Form of expression: refers to the form of language that people prefer to use to refer to objects.
- Indicators for each of the modes of thinking in each factor are proposed below.

Table 2 - Indicators for factors used in the characterization of modes of thinking.

Factors	**Modes of thinking**	
	Synthetic-Geometric	**Analytical-Arithmetic**
Relationship with the object	The object is given directly.	The object is given indirectly.
Action or claim	Concrete. Attempts to describe the object. Visualizes positions in space. Visualizes relationships between objects (vectors, lines, planes). Visualizes all possible cases (e.g., of lines in three-dimensional space). Graphically represents possible solutions to a system of equations.	Abstraction. Tries to find possible solutions (forms of organization). Simplifies calculations. Substitutes variables. Uses formulas. Solves systems of equations.
Attitude	Practical.	Theoretical.
Form of expression	Language used refers to figures, graphic representations. Direct.	Figures are understood as sets ~~with~~ fulfilling certain conditions. Systemic.

Source: Prepared by the authors.

The characterization of the manifestation of students' mathematical thinking in problem solving was carried out by means of a rubric, which rather than serving to qualify the solution of individual problems, served to characterize the forms and levels of a student's mathematical thinking based on a sufficiently large number of problems solved. This rubric is an original proposal of the present research and can be seen in the following table.

Table 3 - Rubric for observation of the development of mathematical thinking.

Dimension	Low level	Medium level	High level
Changes in the representation of mathematical objects	Does not establish relationships between ways of thinking	Establishes relationships between arithmetic and geometric modes of thinking	Establishes relationships between arithmetic, geometric and structural modes of thinking.
Modeling of problems using mathematical objects	Does not demonstrate knowledge of the characteristics of mathematical objects.	Demonstrates knowledge of the characteristics of mathematical objects but does not use them effectively to model problems	Models problems using pertinent mathematical objects
Formulation of problem-solving strategies	Does not propose or proposes unconnected ideas concerning problem-solving strategy	Proposes complete strategies to solve problems	Proposes insightful or diverse strategies for solving problems
Development of problem-solving strategies	Has difficulties following previously formulated problem-solving strategies	Makes minimal errors when proposing problem-solving strategies	Fully develops problem-solving strategies
Communication of mathematical problem solving	Has difficulties in explaining problem solving attempts	Clearly explains problem solving	Argues for Fully explains problem solving strategy used
Structuring of thought	The evolution of actions in processes is evidenced	The encapsulation of processes in objects is evidenced.	The generation of thinking schemes is evidenced

Source: Prepared by the authors.

Just as this rubric was used to characterize students' performance in solving the problems proposed in the adaptive curriculum, a survey was designed to collect information regarding their perception of the curriculum and the ALMS. Another similar survey was applied to the teachers who were in charge of the first iteration. Based on the results of this survey, a process of reflection was carried out and adjustments were made to the curriculum after each iteration. These adjustments involved changes for the second iteration, such as:

- Working groups were made up of two or three students, i.e., the work was not individual. This was done so that communication among students would enhance the development of mathematical thinking.
- When the students solved the problems, they could continue advancing through the node without having to wait for express authorization from the teacher, so they would have smoother progress and better use of time.
- The number of problems students had to solve in each node was reduced.
- An indicator with the number of problems to be solved was added when the node was introduced.
- In addition to the explanations in text format provided during the moments of theoretical development, video explanations were added.

ANALYSIS OF RESULTS

A first element to be highlighted in the results of this research is related with the results of the test of characterization of the modes of thinking applied to the participating students.

Table 4 - Distribution of students according to modes of thinking.

Iteration	Synthetic-Geometric	Analytical-Arithmetic
1	47.6%	52.4%
2	40.6%	59.4%

Source: Prepared by the authors.

The table above shows the distribution of students according to the modes of thinking (synthetic-geometric and analytic-arithmetic) in each of the two iterations of implementation of the adaptive curriculum. The percentages in both iterations are quite similar in distribution, however, as might be expected given the way mathematics is taught in Colombian schools, the analytical-arithmetic mode of thinking prevails among students. As a note, the test did not discriminate students according to the analytical-structural mode of thinking, because it was considered that this mode of thinking is not developed in secondary school in Colombia.

A second element that is important to highlight in these results shows the perception of students and teachers with respect to the curriculum and the ALMS. The survey measured the following for each of the stakeholders.

- For students
    - Platform: ease of navigation and aesthetics.
    - Methodology: clarity of teaching, quality of the different moments singled out by the system and their adaptive features.
    - Temporal factors: duration of the module and intensity of the work.
    - Perception of learning: “feeling” when learning autonomously and adaptively.
- For teachers
    - Platform: ease of navigation, aesthetics, and functionality.
    - Methodology: adaptive features.
    - Temporal factors: program duration and intensity of work.

Table 5 - Comparative results of the perception survey applied to students between iterations (data in percentages).

	Iteration 1 (%)	Iteration 2 (%)
1. The way the learning moments are presented on the platform.		
It facilitated the reading of the learning contents.	13.9	9.5
It was complicated at times.	61.1	61.9
It was difficult to understand.	25.0	28.6
2. Navigation on the platform		
It allowed easy access to information.	47.2	66.7
At times it made it difficult to access information.	38.9	20.6
It was confusing.	13.9	12.7
3. The information presented		
It was clear in general.	16.7	19.0
It required being complemented by the teacher’s explanation.	50.0	44.4
It was difficult to understand even with explanation.	33.3	36.5

4. In general, how did you find the level of difficulty of the comprehension questions and exercises?		
Easy	8.3	3.2
Adequate	33.3	46.0
Difficult	44.4	38.1
Very difficult	13.9	11.1
5. In general, how did you find the level of difficulty of the problems?		
Easy	0.0	0.0
Adequate	19.4	20.6
Difficult	66.7	63.5
Very difficult	13.9	15.9
6. Regarding your motivation during the learning process		
It always remained high, no matter what content or moments I was in.	5.6	9.5
It increased as I progressed and understood the dynamics better.	8.3	34.9
It declined as progress was made and there was an increase in difficulty.	66.7	42.9
It was low during the process.	19.4	12.7
7. Regarding the interaction with the teacher in the process of solving the problems you consider that		
It was required in the feedback of problems.	55.6	57.1
Other than on problems, it was required on more occasions.	33.3	28.6
It made no difference.	11.1	14.3
8. Regarding the interaction with your classmates during problem solving, do you consider that it		
Enriched the consolidation of learning.	38.9	50.8
Sometimes obstructed or delayed the learning process.	36.1	38.1
Was not necessary and could be a completely individual process.	25.0	11.1
9. The time given for learning in the subject module was		
Sufficient to achieve the personal learning expected by you.	44.4	42.9
Insufficient to achieve your personal learning expectations.	55.6	57.1
10. The amount of time you invested in the development of the module was		
Higher than what I am used to in this area.	47.2	74.6
Similar to what I am used to in this subject.	36.1	20.6
Lower than what I am used to in this matter.	16.7	4.8
11. This way of learning seemed to you		
Not as good as the traditional way.	77.8	11.1
As good as the traditional way.	16.7	23.8
Better than the traditional way.	5.6	65.1
12. During your learning process, did you feel identified with the mode of thinking assigned to you by the adaptive system?		
All the time	2.8	7.9
Most of the time.	52.8	28.6
Some of the time.	27.8	49.2
A few times	11.1	9.5
Never	5.6	4.8
13. Classification in the mode of thinking		
It facilitated my learning process.	41.7	38.1
It was indifferent to my learning process.	41.7	42.9
It hindered my learning process.	16.7	19.0

Source: Prepared by the authors.

Although there is not sufficient space in this article to analyze in detail the answer to each of these questions, it is interesting to highlight some specific points. For example, regarding question 1, in the first iteration it can be observed that a high percentage of students think that the presentation of the learning content was difficult to understand at least in some moments, this may be because students have poor reading skills, especially in reading mathematical texts. To address this weakness of the curriculum in the second iteration, video explanations were added to the text explanations, which however did not cause a difference in perception (although it should be noted that the groups in the two iterations had different students).

Regarding question 6, the fact that a large proportion of the students indicated that motivation was low implies that the challenge problems probably did not generate commitment in them. However, this could also be due to the way the lessons were presented (in writing) and the fact that they had to work autonomously, without the teacher directly guiding the process. In any case, it can be seen that in the second iteration there was an increase in the perception of motivation, and although it is not clear what the causal factor is, it could be speculated that the fact that the students worked in groups helped them to stay motivated, given what is seen in question 8 in which the report that working in groups enriched the process was significantly higher in the second iteration than in the first.

Regarding the interaction with the teacher in the process, many of the students value as necessary the interaction with the teacher for feedback on the problems and a good part of the students refer to it as necessary at other times as well.

The response of the students who participated in the first iteration to question 11 on the comparison between this way of learning and the traditional way of learning is very interesting. At the time the question was asked, the researcher intended for students to compare adaptive learning and learning according to the single model, however, it is likely that other factors may have had a greater impact on the response than initially thought. Factors such as: interacting with a system where learning content is presented in written form; the absence of a teacher with the role of explaining content when first encountered, where "explaining" means presenting examples that they could then replicate; having to solve non-routine problems, which is not a common methodology in mathematics teaching at the school or university level; or having to work in groups on some problems, which made it difficult for some students to progress at a rapid pace. These factors are inherent to this curriculum design because it not only contemplates the design of an adaptive platform, but also of a curriculum applicable in adaptive environments. Considering this, in the second iteration some modifications were made to mitigate the influence of the factors described above: the difficulty of the non-routine problems presented to the students was reduced and working groups were formed from the beginning, which is clearly reflected in their perception of the experience.

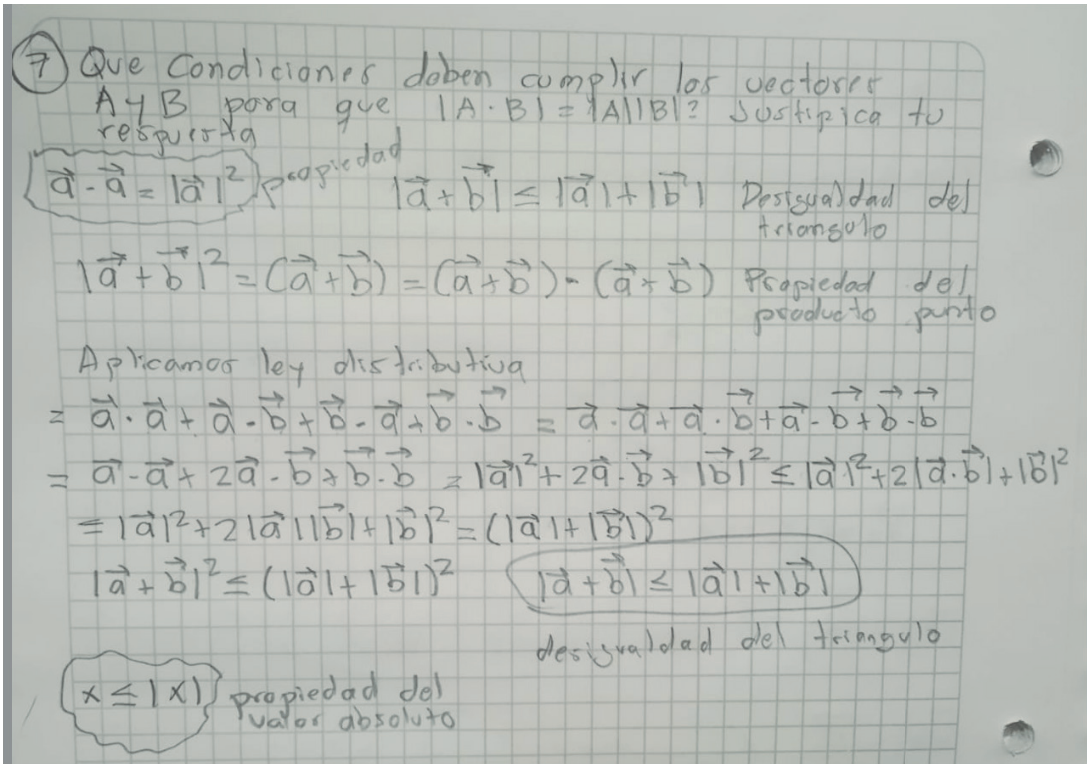
Based on how it was answered, another important question is number 12 in which it can be observed that the students participating in the first iteration felt identified, for the most part, with the mode of thinking in which they were classified by the characterization test. This contrasts with the answers to question 13 in that same iteration in which less than half of the students indicate that having been placed according to their thinking mode was convenient for the learning process, although it could also be considered that the students who answered indicating that it was indifferent for their process did not note that it was detrimental to them, so they could be added to the previous percentage. In the second iteration the number of students choosing the first three options is very similar to that of the first iteration, although its distribution is different, since more students in the second iteration think that the grading system was only relevant sometimes, which is still paradoxical since it does not correspond with the answer to the last question in which a high percentage (38.1%) points out that the curriculum facilitated their learning process.

Finally, the third aspect to be singled out in these results is related to evidence of students' mathematical thinking. This was identified by analyzing how students solved the challenging problems. Such analysis was done based on the rubric presented earlier in this article. The challenging problems were proposed to the students depending on the mode of thinking in which they were classified, this did not imply that students who were classified, for example, in the synthetic-geometric mode of thinking were only proposed problems of that type, but it did imply the order (and quantity) in which the problems were presented. That is, first problems related to the mode of thinking in which the student was classified were presented and then changes of representation to the other modes of thinking were proposed as they progressed through each of the nodes. This was done because changes of representation between modes of thinking are a determining factor in the development of mathematical thinking, as is explicit in the first dimension of the rubric.

In addition to this first characteristic manifestation of mathematical thinking, four other characteristics were detected in the review of the solutions that students proposed to the problems.

Problems of modeling mathematical objects was one of them. When confronted with problems, most students were able to identify the mathematical object that modeled the situation. This is an initial step in the application of mathematical thinking, which does not necessarily imply that the problems are correctly solved. In this case, the following figure shows an example of a group of students who did not correctly model a problem.

Figure 4 - Solution of problem 7 of node 3 of the analytic-arithmetic mode of thinking.



Source: Prepared by the authors.

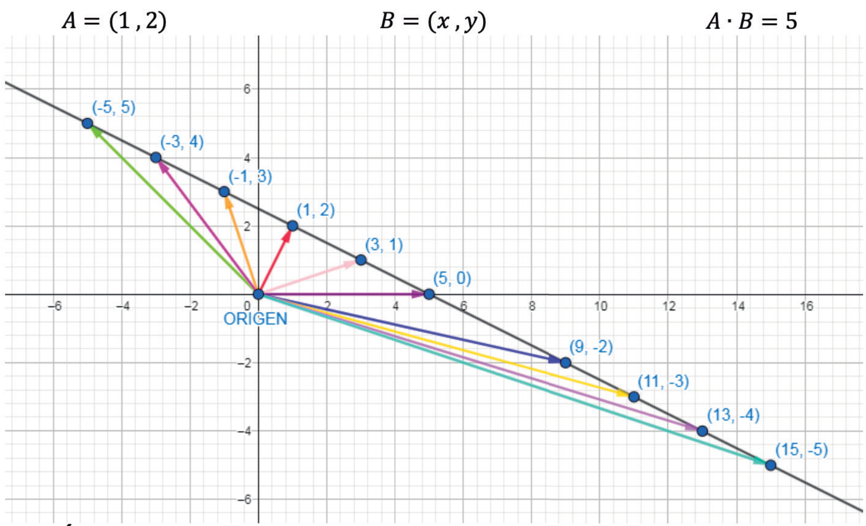
In this solution the students were supposed to model the problem using the geometric presentation of the dot product, however they make use of the triangular inequality and the property of the dot product of a vector with itself.

Mathematical thinking is also evident in the correct formulation of strategies to solve problems. This is something that is difficult for students to do when faced with challenging problems (when the strategy is not obvious), because they are highly accustomed to solving routine problems for which they are familiar with the strategy. For example, in the following problem students correctly model the problem, but do not use a good strategy. In general, in this problem, students searched for vectors with coordinates in the integer grid, which resulted in a correct solution of the problem, but not with a correct strategy.

Figure 5 - Solution of problem 11 of node 2 analytic-arithmetic mode of thinking.

PROBLEMA #11

Problema 11. *Si* $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, *considera todos los vectores* $B = \begin{bmatrix} x \\ y \end{bmatrix}$ *en el plano* xy *tales que* $A \cdot B = 5$. *¿Cuál de todos estos vectores* B *es el más corto?*



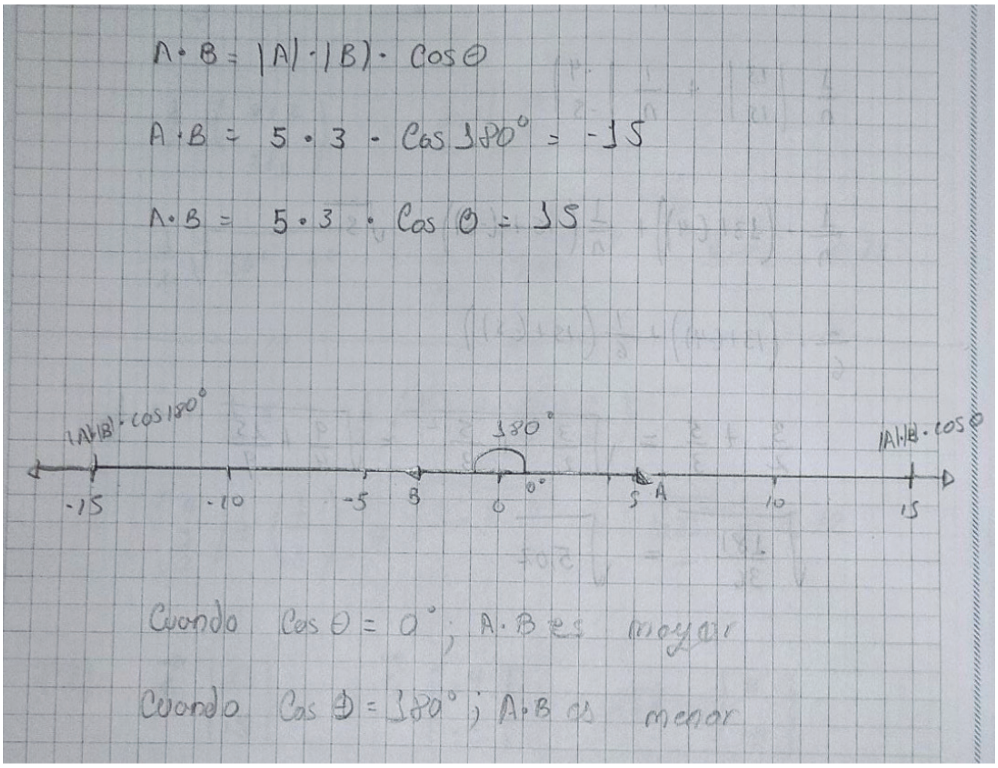
JUSTIFIACIÓN:

De acuerdo a la gráfica anterior, se considera que el vector más corto es B= (1,2).

Source: Prepared by the authors.

Correct communication of the solution of mathematical problems is a characteristic with which mathematical thinking can be evidenced; at first glance this would seem not to be so relevant, but once a pattern of regularity in a student's performance with respect to this factor is observed, it can also be noticed that their mathematical thinking has developed at a good level. Reciprocally, in some cases students solve the problems, but the communication of the solution is poor, from which it is probably correct to infer that, although they know the methods, they do not have a clear idea of the purpose of what they are doing.

Figure 6 - Solution of problem 1 of node 2 synthetic-geometric mode of thinking.



Source: Prepared by the authors.

In the previous problem, the students had to find the largest and smallest value of the expression in the first line, considering that y . As can be noticed, already in the third line the problem was solved, however, they make a change of representation, very much in accordance with their mode of thinking, and communicate the answer in a more complete way by making an arithmetic analysis of the geometric form presented.

Finally, and in line with the APOS theory, it is possible to note different levels in the structuring of mathematical thinking; one level in which actions, which are constructed when repeatedly given responses to stimuli, evolve into processes through the internalization of actions; another level in which objects are constructed by encapsulating processes; and a level in which actions, processes and objects are abstracted to give rise to schemas.

Problem 8 of node 1 of the arithmetic mode of thinking was proposed in order to evidence the use of schemes by students, in this case the scheme generated by the dot product. In the following image the statement of the problem can be seen.

Figure 7 - Problem 8 of node 1 analytic-arithmetic mode of thinking.

Problema 8. *In physics, the law of levers is expressed as follows:* $P \times B_P = R \times B_R$*, where* P *means the force applied (power),* R *means the force or weight to be moved (resistance),* B_P *is the distance from the fulcrum to the point of application of the power and* B_R *is the distance from the fulcrum to the point of location of the resistance, as can be seen in the following image.*

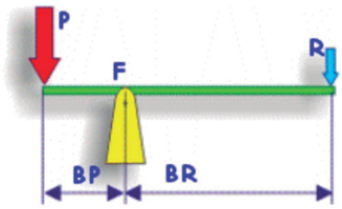


Figura 9: Elements of a lever

Use the dot product to model this law.

Source: Prepared by the authors.

This problem was particularly difficult for most of the students, since having a scheme implies recognizing its parts and being able to use them properly, i.e., it is required to recognize which objects are involved and which processes with those objects should be used.

CONCLUSIONS

The research reported in this article explored various learning conditions from which it is possible to adapt a curriculum, and the decision was made to use the modes of thinking proposed by Sierpinska (2000), since they are relevant to the fundamental purpose of the proposed curriculum, which is the development of mathematical thinking, and are suitable for modeling the learning of the contents of linear algebra. The modes of thinking do not depend only on the cognitive forms of the students, but also on the epistemological form of the learning content.

To classify students in the modes of thinking, a test was designed to determine whether a student is more inclined towards the synthetic-geometric mode of thinking or towards the analytical-arithmetic mode. The application of this test showed that, contrary to what could be hypothesized (that there would be a majority inclination towards the analytical-arithmetic mode of thinking), students are quite homogeneously distributed between these two categories.

To structure the learning content, the theoretical framework related to the genetic decomposition of the APOS model of Dubinsky (1991) was used. This model was used taking into account not only the epistemological characteristics of the learning contents, but also the characteristics of the students' modes of thinking.

The proposed methodology integrates important novelties with respect to what is known in the state of the art. These novelties are:

- Unlike the adaptive curricula reviewed, which base their adaptation and outcomes on student performance, the curriculum proposed in this research proposes a framework for the characterization of students' mathematical thinking and uses adaptive factors associated with this.

- An active role for the teacher is proposed, i.e., while several of the adaptive systems reviewed seek automation and, therefore, the exclusion of the teacher, the system proposed in this research assigns a role to the teacher in the teaching-learning process.
- The adaptive systems reviewed privilege individual work, however, the curriculum proposed in this research recognizes the importance of communication among students for the development of their mathematical thinking.
- The adaptive curriculum proposed in this thesis uses as a resource for the development of mathematical thinking the resolution of challenging problems. In addition, the document that records the research process presents the result of an inductive analysis of the characteristics of this type of problems.

In this research, a rubric was proposed to detect students' mathematical thinking, particularly when solving challenging problems. Based on what was reported in this thesis, it can be seen that students put into play, at different levels, their mathematical thinking when solving these problems.

The context of the proposed methodology for the design of adaptive curricula is limited to mathematics subjects, but it could be extrapolated to subjects in other disciplinary fields, given the importance and relevance of adaptive education.

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REFERENCES

ARNON, I, et al. **APOS theory:** A framework for research and curriculum development in mathematics education. 2014.

BETTAHI, J. **Personalized Learning:** One-Size-Fits-One Model. 2018.

BROWN, A. L. **Design experiments:** Theoretical and methodological challenges in creating complex interventions in classroom settings. Journal of the Learning Sciences, v. 2, n. 2, p. 141-178. 1992.

BRUSILOVSKY, P. Adaptive hypermedia: From intelligent tutoring systems to web-based education. In: GAUTHIER, G.; Frasson, C.; VANLEHN, K. (Eds.). **International Conference on Intelligent Tutoring Systems**. Springer, Berlin, Heidelberg, 2000. p. 1-7.

DEMBO, M.; HOWARD, K. **Advice about the use of learning styles**: A major myth in education. Journal of college reading and learning, v. 37, n. 2, p. 101-109, 2007.

DUBINSKY, E; LEWIN, P. **Reflective abstraction and mathematics education:** The genetic decomposition of induction and compactness. The Journal of mathematical behavior, v. 5, n. 1, p. 55-92, 1986.

DUNN, R.; DUNN, K.; PRICE, G. **Learning style inventory**. Lawrence, KS: Price Systems, 1981.

FELDER, R.; SILVERMAN, L. **Learning and teaching styles in engineering education**. Engineering education, v. 78, n. 7, p. 674-681. 1988.

GRAF, S. et al. Adaptivity and personalization in learning systems based on students' characteristics and context. In: **The 1st international symposium on smart learning environment**. 2012. p. 33-36.

HONEY, P.; MUMFORD, A. **Learning styles questionnaire**. Organization Design and Development, Incorporated, 1989.

KEEFE, James W. **Learning Style Theory and Practice**. National Association of Secondary School Principals, 1904 Association Dr., Reston, VA 22091, 1987.

KIRSCHNER, P. **Stop propagating the learning styles myth**. Computers & Education, v. 106, p. 166-171, 2017.

KOLB, David A. Learning styles inventory. **The Power of the 2**, v. 2, p. 267. 1985.

MUDRÁK, M. **Personalized e-course implementation in university environment**. International Journal of Information and Communication Technologies in Education, v. 7, n. 2, p. 17-29. 2018.

MURRAY, M.; PÉREZ, J. **Informing and performing: A study comparing adaptive learning to traditional learning.** Informing Science: The International Journal of an Emerging Transdicipline, v. 18, p. 111, 2015.

MYERS, I.; MCCAULLEY, M.; MOST, R. **Manual, a guide to the development and use of the Myers-Briggs type indicator**. Consulting psychologists press, 1985.

POPESCU, E.; BADICA, C.; MORARET, L. **Accommodating learning styles in an adaptive educational system.** Informatica, v. 34, n. 4. 2010.

POWELL, W.; KUSUMA-POWELL, O. **How to teach now**: Five keys to personalized learning in the global classroom. Ascd, 2011.

ROSÉ, C. et al. **Explanatory learner models:** Why machine learning (alone) is not the answer. British Journal of Educational Technology, v. 50, n. 6, p. 2943-2958, 2019.

SIERPINSKA, A. On some aspects of students' thinking in linear algebra. In: **On the teaching of linear algebra.** Springer, Dordrecht, 2000. p. 209-246.

SKEMP, R. **Relational understanding and instrumental understanding**. Mathematics teaching, v. 77, n. 1, p. 20-26, 1976.

YARANDI, M.; TAWIL, A.; JAHANKHANI, H. Ontologies for personalised adaptive learning. In: **Advances in Computing & Technology**. University of East London, School of Architecture Computing and Engineering, 2012.

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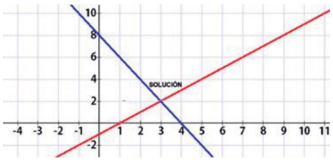
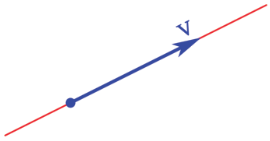
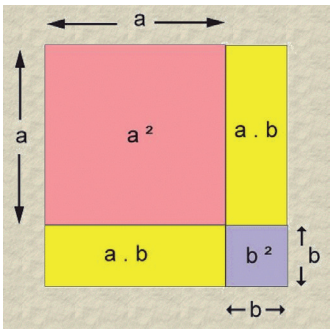
APPENDIX

The questions that made up the test are presented below, arranged according to the four design factors.

Factor: Relationship with the object

1. When faced with a system of equations, I prefer to see its solution as follows:						
	1	2	3	4	5	x = 3 y = 2
2. I prefer to know an object by						
its image.	1	2	3	4	5	its features.
3. When I imagine a vector, I think of						
	1	2	3	4	5	$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
4. When faced with the need to describe a place, I prefer to						
make the scheme or drawing.	1	2	3	4	5	list their characteristics.

Factor: Form of expression

5. To assemble a piece of furniture, I prefer to be guided by						
images of what to do.	1	2	3	4	5	instructions in the manual (no images).
6. To understand that $(a + b)^2 = a^2 + 2ab + b^2$, I prefer						
	1	2	3	4	5	$(a + b)(a + b)$ $= aa + ab + ba + bb$ $= a^2 + 2ab + b^2$
7. I agree more with the phrase:						
"The whole is the sum of the parts"	1	2	3	4	5	"Each part is a whole"
8. In order to know the characteristics of a conic section, I prefer to use its						
graphic.	1	2	3	4	5	equation.

Factor: Action or pretensión

9. When I have to get somewhere new, I prefer to find my way around with						
a map.	1	2	3	4	5	an address.
10. "What is the number that when added to its double gives 24?". To solve this problem, I prefer						
try out the solution.	1	2	3	4	5	formulate equations.
11. When giving directions to a friend to my home, I prefer to						
locate important points of reference.	1	2	3	4	5	set up a route with turns, paths and times.

12. I identify more with someone who builds a construction.						
by intuition.	1	2	3	4	5	based on calculations.

Factor: Attitude

13. To know a story, I prefer						
watch a movie.	1	2	3	4	5	read a book.
14. When setting up a new electronic device, I prefer to						
interact with it.	1	2	3	4	5	read the manual.
15. When faced with a problem, I prefer						
try several possible solutions.	1	2	3	4	5	consult the theory to find the solution.
16. When I have to make a decision, I am guided more by						
my intuitions.	1	2	3	4	5	the judgment of the facts.